Analytical Modeling of Natural Convection in Horizontal Annuli

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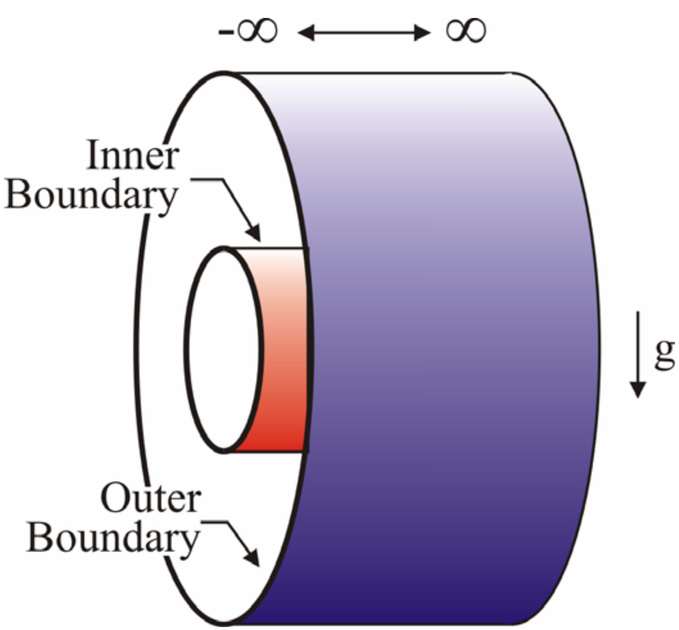
Outline

- Introduction and problem description
- Literature review and objectives
- Model development
- Validation
- Summary and conclusions

Problem Definition

- 2D horizontal annulus
- Steady state, natural convection
- Concentric inner and outer cylinders
- Isothermal boundary conditions, $T_i > T_o$



Geometry:

- Relative boundary size

$$P_o / P_i \Rightarrow d_o / d_i \quad \text{(spheres)}$$

- Effective gap spacing

$$\delta_e \Rightarrow (d_o - d_i)/2 \quad \text{(spheres)}$$

- Shape, orientation

Parameter Definitions

- Total heat transfer rate non-dimensionalized by Nusselt number

$$Nu = \frac{Q\,\ell}{k\,P_i\left(T_i - T_o\right)}$$

- P_i selected as characteristic length:
  - For $P_o / P_i \rightarrow \infty$ limit, scale length related to inner body dimensions only
  - Similar results for similar body shapes, orientations

$$Nu_{P_i} = \frac{Q}{k\left(T_i - T_o\right)}$$

Parameter Definitions

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- Dimensionless conduction shape factor

$$\lim_{Ra \ll Ra_{cr}} Nu_{P_i} = S^*_{P_i}$$

- Effective conductivity

$$\frac{k_{eff}}{k} = \frac{Nu_{P_i}}{S^*_{P_i}}, \qquad \frac{k_{eff}}{k} \geq 1$$

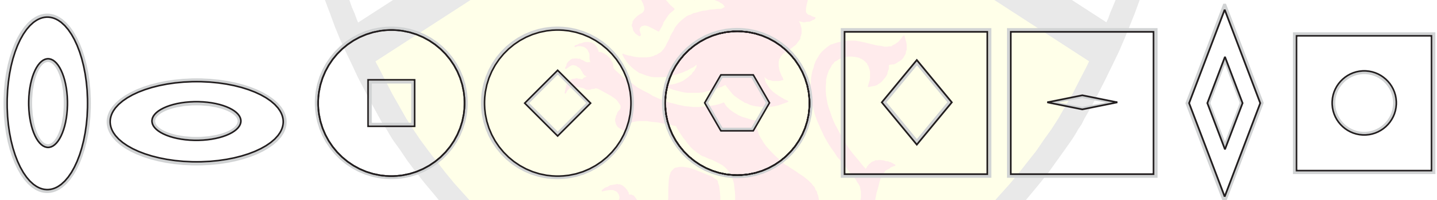
- Rayleigh number

$$Ra_{P_i} = \frac{g\beta\left(T_i - T_o\right)\left(P_i\right)^3}{\upsilon\alpha}$$

Literature Review - Data

Experimental and Numerical Studies

- Concentric spherical enclosures
  - Over 20 publications with average heat transfer data
  - Most experimental data for high Rayleigh number, boundary layer flow
  - All other data from numerical simulations
- Other enclosure geometries
  - Numerical data for circular, polygonal, rhombic, elliptical cylinders



- Correlations of experimental, numerical data
  - Valid for limited ranges of Rayleigh number
  - Geometry-dependent

Literature Review - Models

- Analytical models available for concentric, eccentric circular annulus
  - Raithby & Hollands[28]
  - Kuehn and Goldstein[29]
- Boyd[30] presents general correlation procedure for 2D annulus with arbitrarily-shaped boundaries
  - Requires correlation coefficient values from empirical data
  - Difficult to implement for non-standard boundary shapes

Objectives

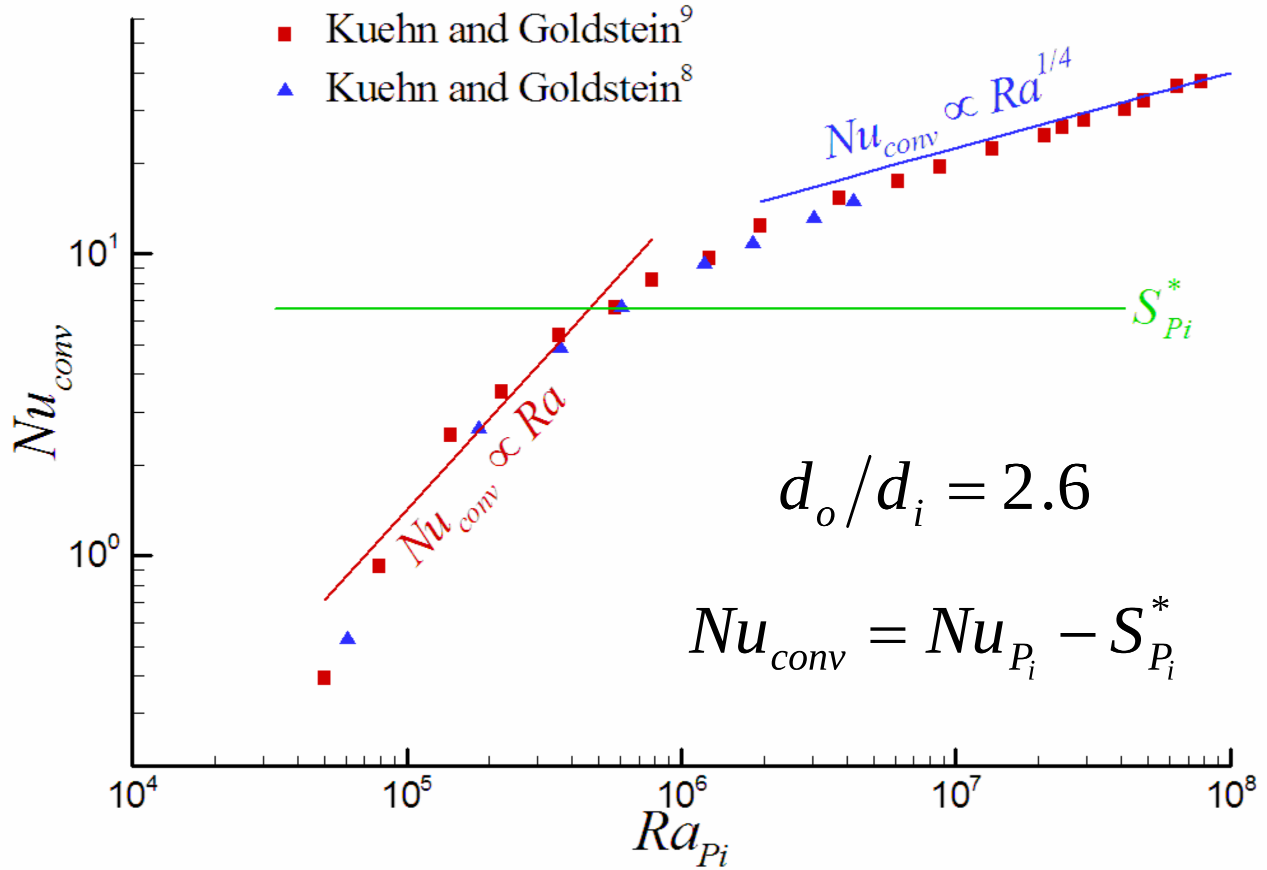
- Analytical modeling of natural convection in horizontal annulus
  - Full range of Ra_{P_i} from conduction to convection
  - Applicable to wide range of geometries
    - Inner and outer boundary shapes and orientation
    - Relative boundary sizes
  - Physically-based analysis
- Validate model using experimental, numerical data from the literature
  - Circular annulus
  - Annuli with different inner, outer boundary shapes

Model Development

- Assume linear superposition of diffusive, convective limits

$$Nu_{P_i} = S^*_{P_i} + Nu_{\text{conv}}$$

- Kuehn and Goldstein[8,9] data for circular annulus



Model Development

- General model based on Churchill and Usagi[33] composite solution technique

$$Nu_{P_i} = S^*_{P_i} + \left[\left(\frac{1}{Nu_{tr}} \right)^n + \left(\frac{1}{Nu_{bl}} \right)^n \right]^{-1/n}$$

- Combination of three asymptotic solutions

$S^*_{P_i}$ = conduction shape factor

Nu_{tr} = transition flow convection

Nu_{bl} = laminar boundary layer convection

- Combination parameter *n* determined from validation with numerical, experimental data

Conduction Shape Factor

- Correlations, models, from handbooks
- Numerical simulations
- Approximate method from equivalent circular annulus

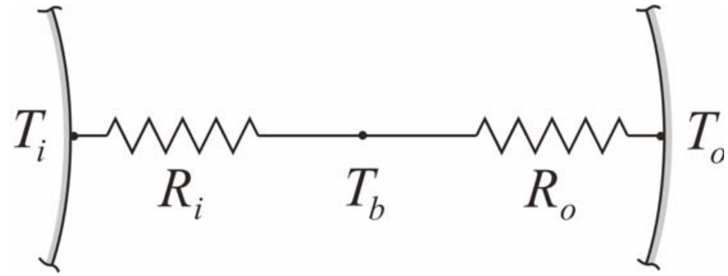
$$S^*_{P_i} = \frac{2\pi}{\ln(d_o/d_i)_e}$$

  - Effective diameter ratio $\left(\frac{d_o}{d_i}\right)_e \Rightarrow$ Inner perimeter $d_i = P_i/\pi$, Enclosed area $d_o = \sqrt{\frac{4A}{\pi} + \frac{P_i^2}{\pi^2}}$
  - Dimensionless conduction shape factor

$$S^*_{P_i} = \frac{2\sqrt{\pi}}{\ln\sqrt{4\pi\left(A/P_i^2\right)+1}}$$

Boundary Layer Convection

- Assumptions
  - Laminar flow
  - T_b uniform
  - Non-intersecting boundary layers



- Series combination of resistances

$$R = R_i + R_o \qquad R_i = \frac{T_i - T_b}{Q} \qquad R_o = \frac{T_b - T_o}{Q}$$

- Non-dimensionalize using Nusselt number

$$R_i = \frac{T_i - T_b}{Q} \qquad R_o = \frac{T_b - T_o}{Q} \qquad Nu_{bl} = \frac{1}{k\left(R_i + R_o\right)} = \frac{Nu_i}{1 + 1/\phi}$$

$$\phi = \frac{T_i - T_b}{T_b - T_o} = \frac{R_i}{R_o} = \frac{Nu_o}{Nu_i}$$

Boundary Layer Convection

- Convection modeled using Yovanovich[31] and Jafarpur[36]

$$Nu_{\sqrt{A}} = F(\Pr)\, G_{\sqrt{A}}\, Ra_{\sqrt{A}}^{1/4}$$

- Laminar boundary layer convection asymptote

$$Nu_{bl} = \frac{Nu_i}{1 + 1/\phi} = \frac{F(\Pr)\, G_{P_i}\, Ra_{P_i}^{1/4}}{\left(1 + 1/\phi\right)^{5/4}}$$

$$Nu_{bl} = \frac{F(\Pr)\, G_{P_i}\, Ra_{P_i}^{1/4}}{\left[1 + \left(P_i / P_o\right)^{3/5} \left(G_{P_i} / G_{P_i}\right)^{4/5}\right]^{5/4}}$$

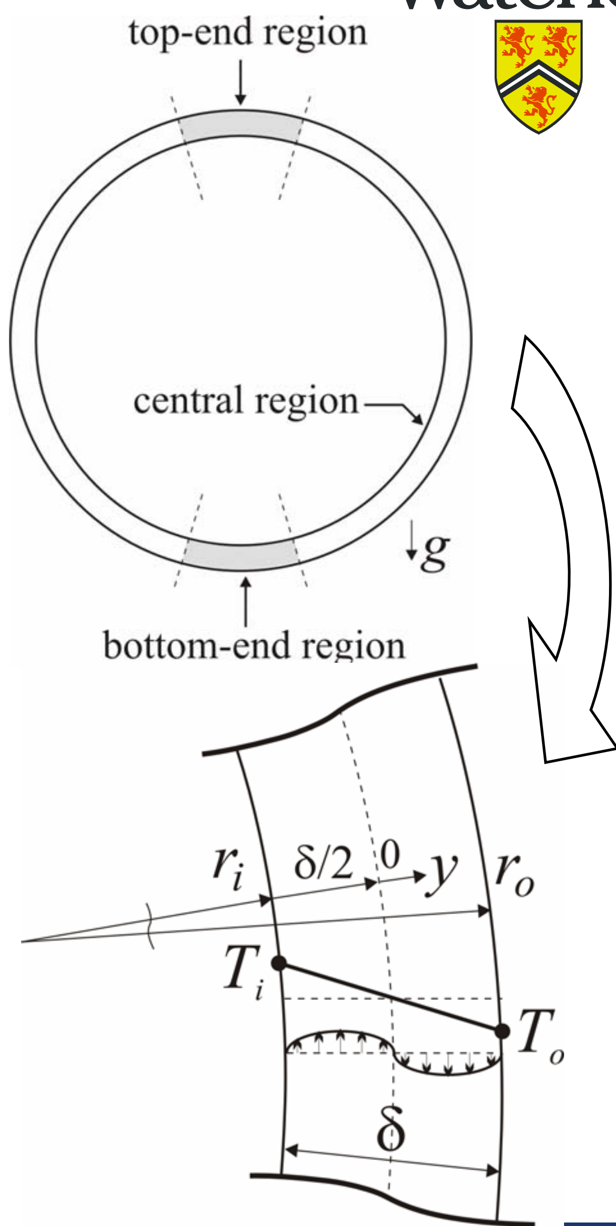
$$Nu_{bl} = \frac{(1.028)\, F(\Pr)\, Ra_{P_i}^{1/4}}{\left[1 + \left(d_i / d_o\right)^{3/5}\right]^{5/4}} \quad \text{(circular annulus)}$$

Transition Flow

- Boundary layers merge when $Ra < Ra_{cr}$
- Model as equivalent circular annulus
- Three distinct regions are formed
- Central region
  - Radial conduction
  - Buoyancy induced flow
- For narrow gap spacing, $\delta_e << r_i$, temperature, velocity in central region

$$T - T_b = -\frac{y}{\delta_e/2}\left(T_i - T_b\right), \quad T_b = \frac{T_i + T_o}{2}$$

$$u = \frac{g_e \beta}{12\upsilon}\left(T_i - T_o\right)\left(\frac{\delta_e}{2}\right)^2\left[\left(\frac{y}{\delta_e/2}\right)^3 - \frac{y}{\delta_e/2}\right]$$



Transition Flow

- Enthalpy balance in top-end and bottom-end regions

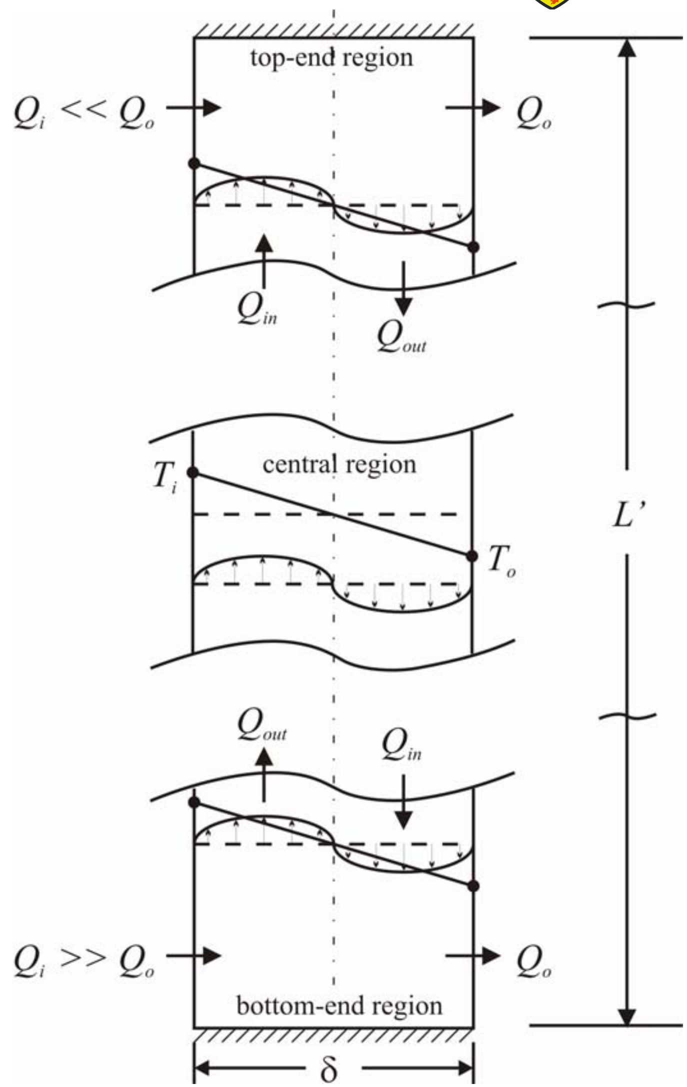
$$Q_{i,o} = \frac{\rho c_p g_e \beta \left(T_i - T_o\right)^2 \delta_e^3}{720 \nu}$$

- Transition flow asymptote

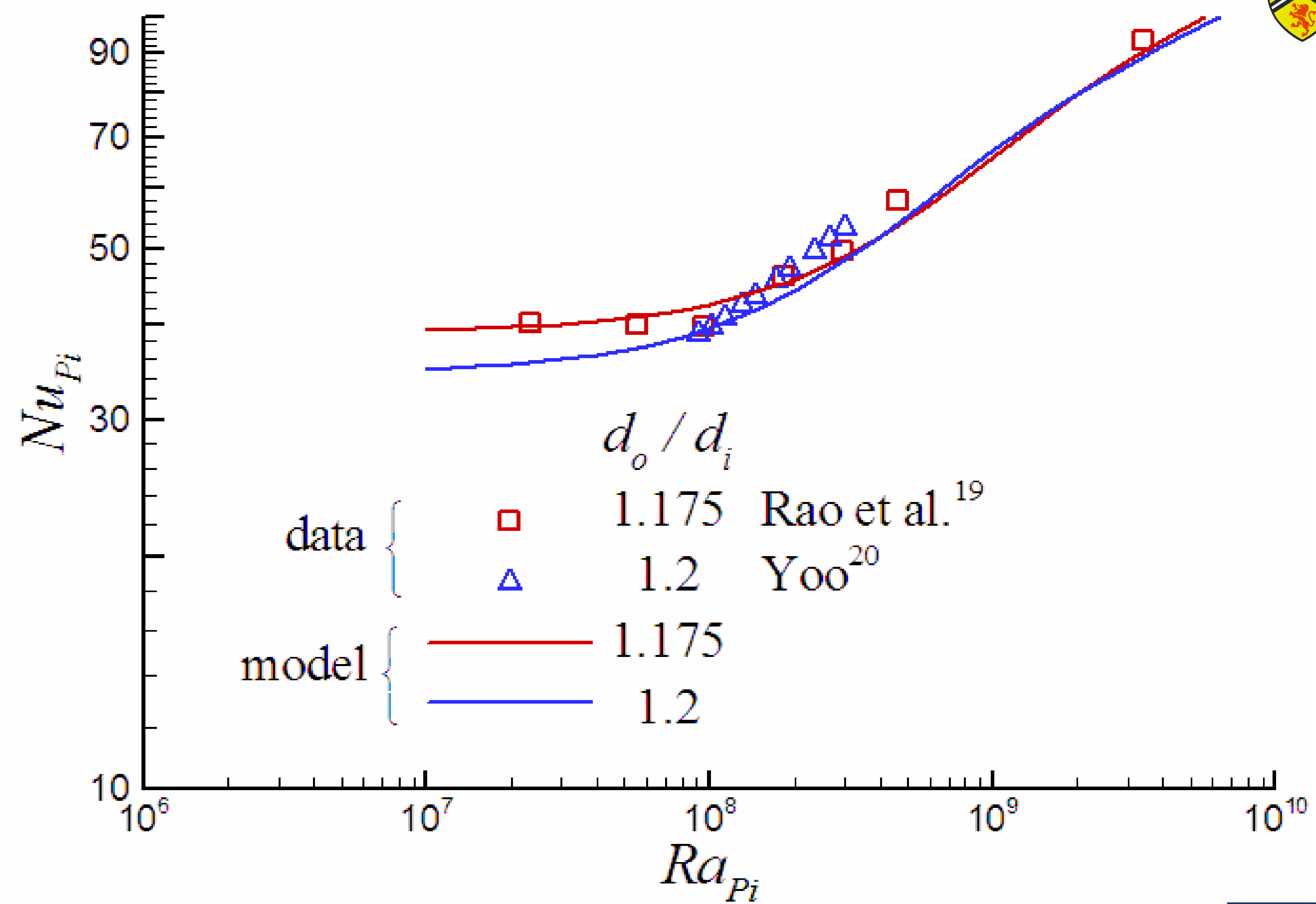
$$Nu_{tr} = \frac{1}{90\pi} \frac{\left(\delta_e / P_i\right)^3}{\left(1 + P_o / P_i\right)} Ra_{P_i}$$

$$Nu_{tr} = \frac{1}{720\pi^4} \frac{\left(d_o / d_i - 1\right)^3}{\left(1 + d_o / d_i\right)} Ra_{P_i}$$

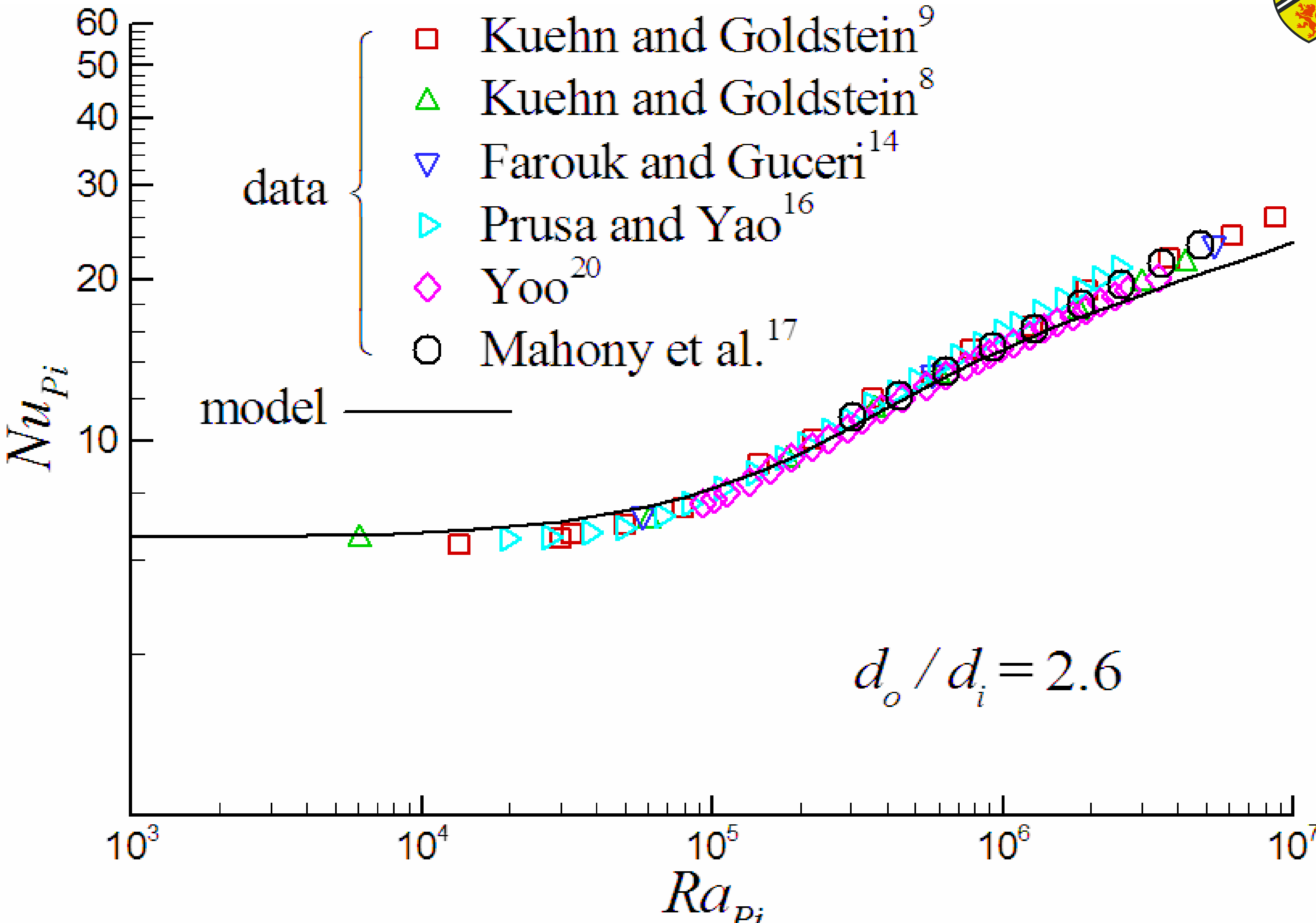
(circular annulus)



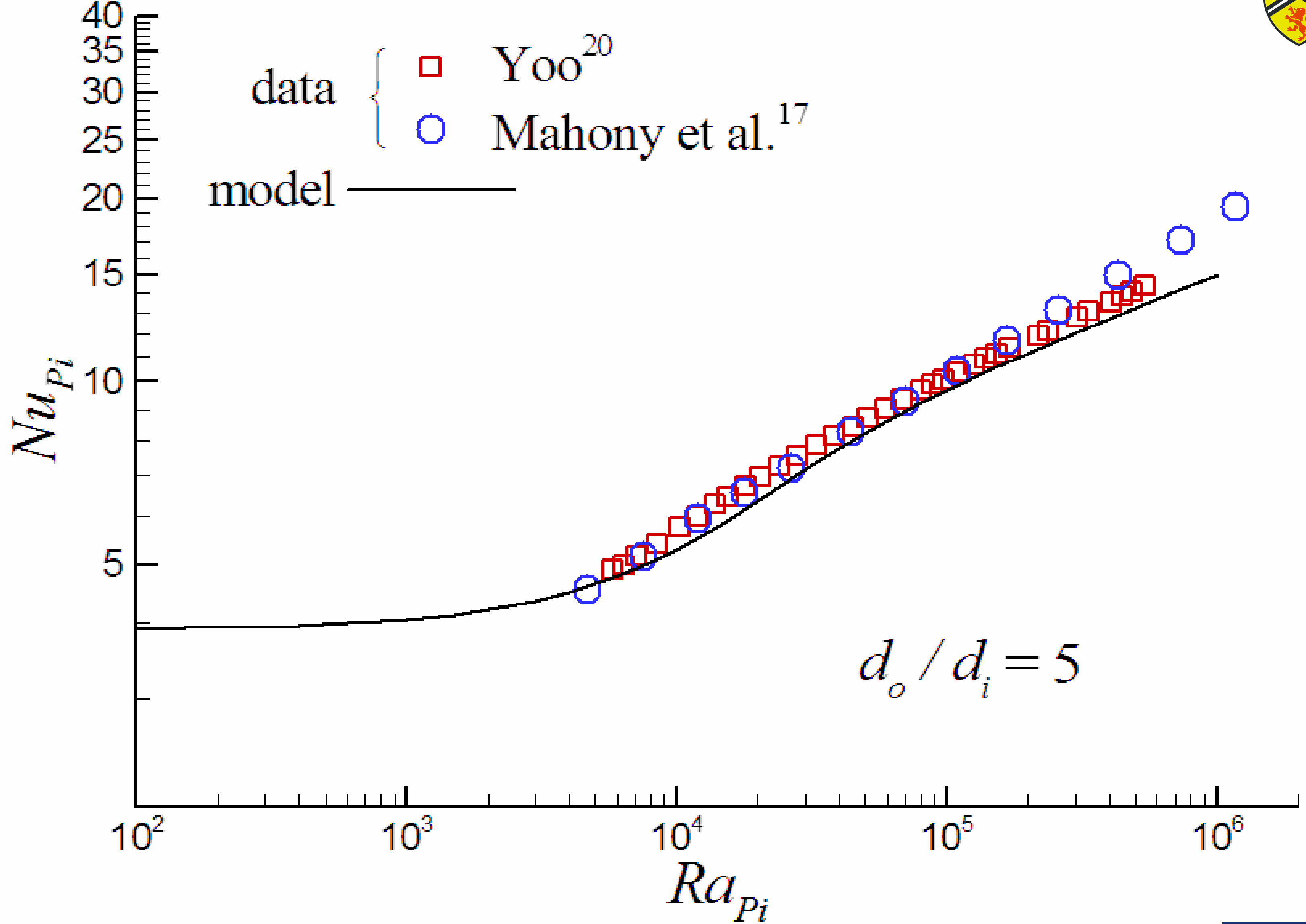
Validation: Circular Annulus



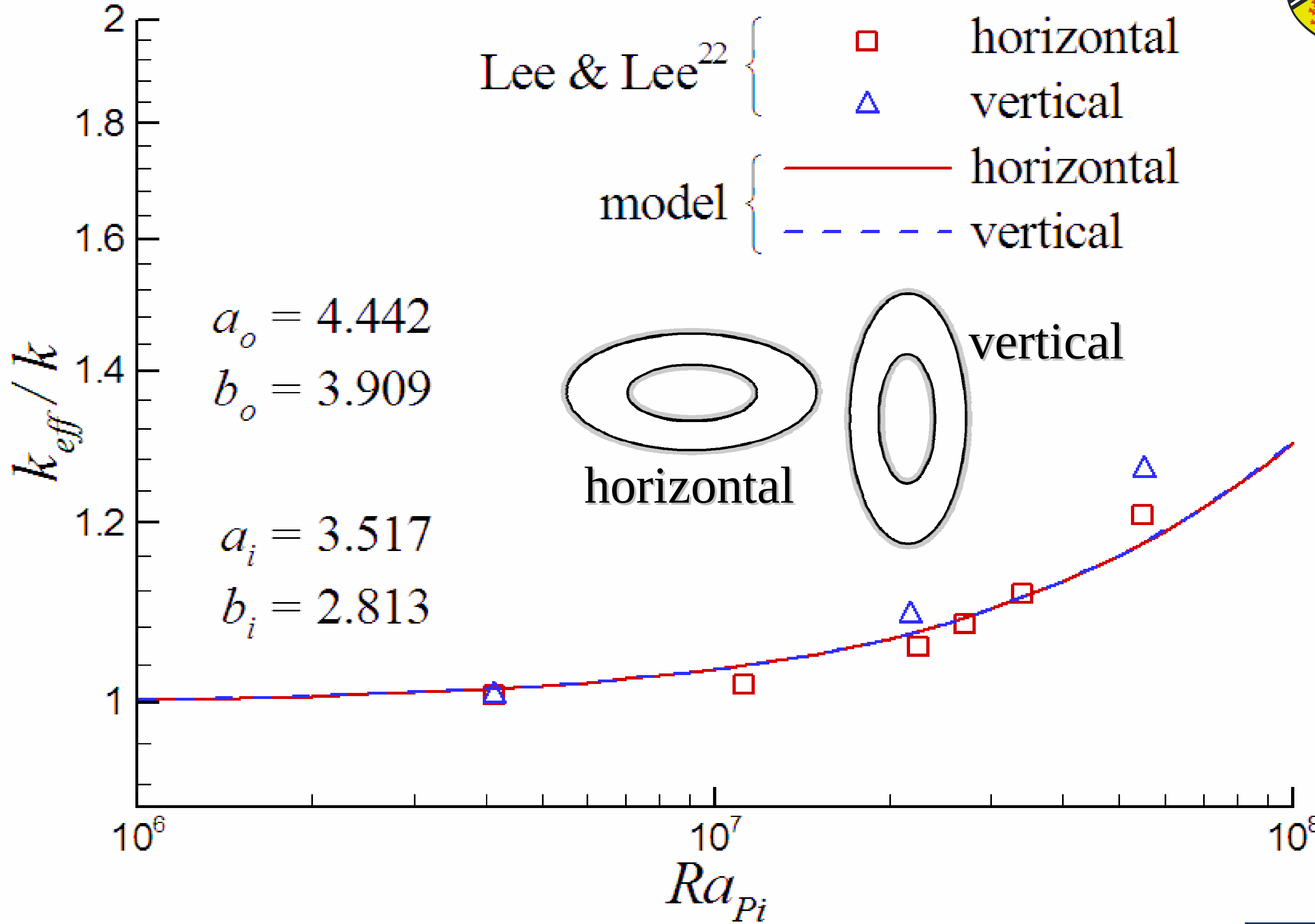
Circular Annulus



Circular Annulus



Concentric Elliptic Cylinders



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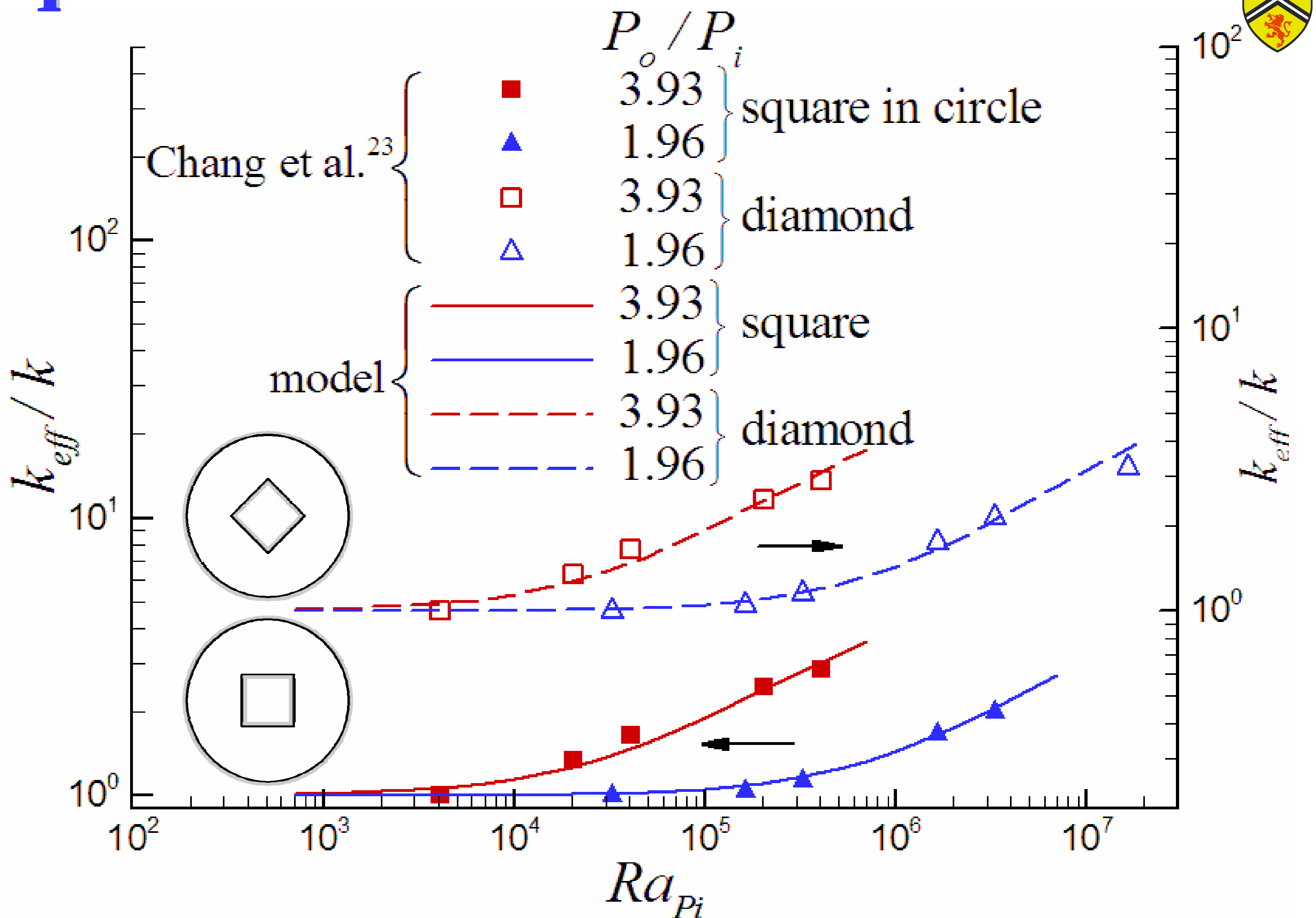
Square and Diamond in Circle



Hexagon in Circular Cylinder

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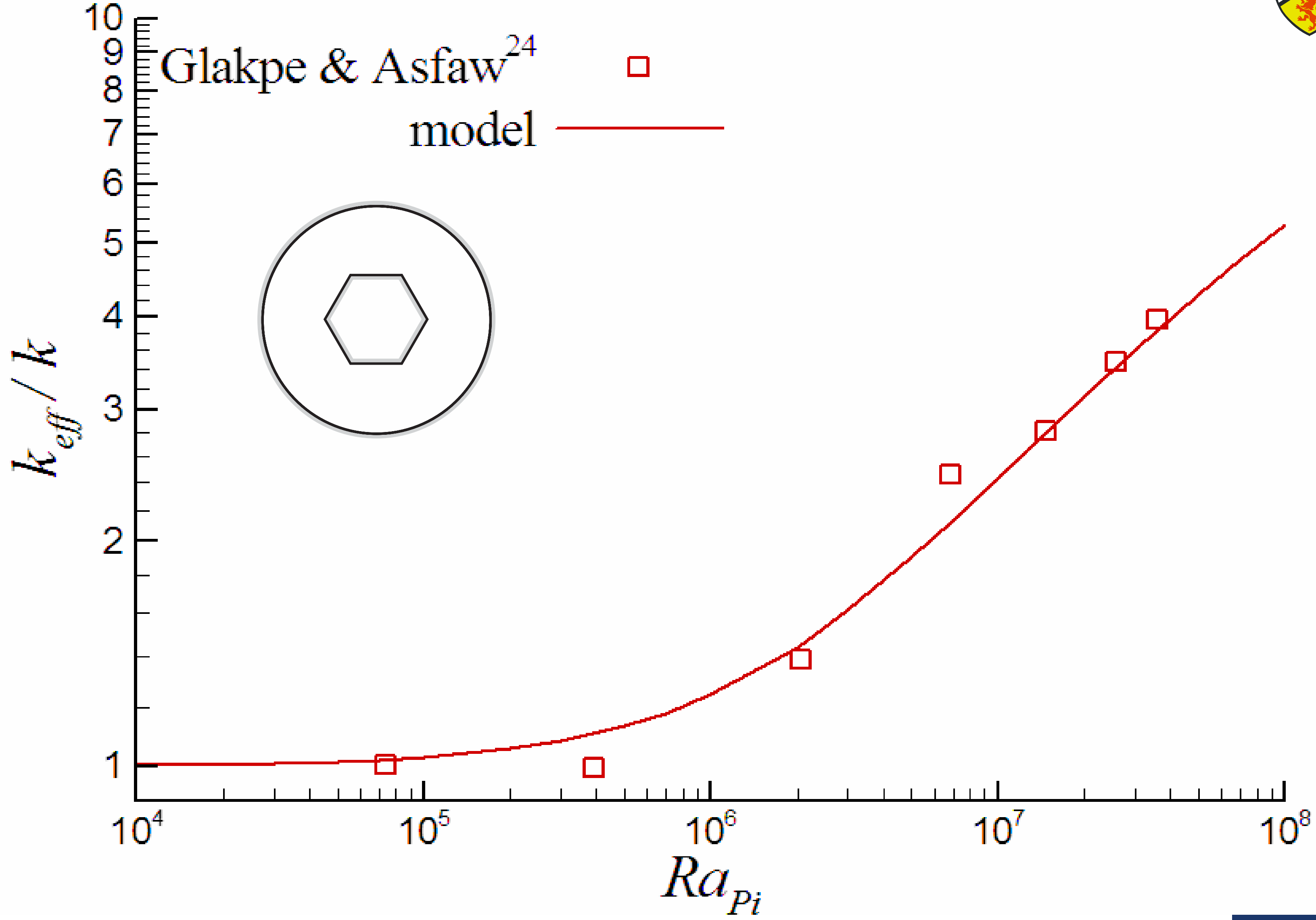

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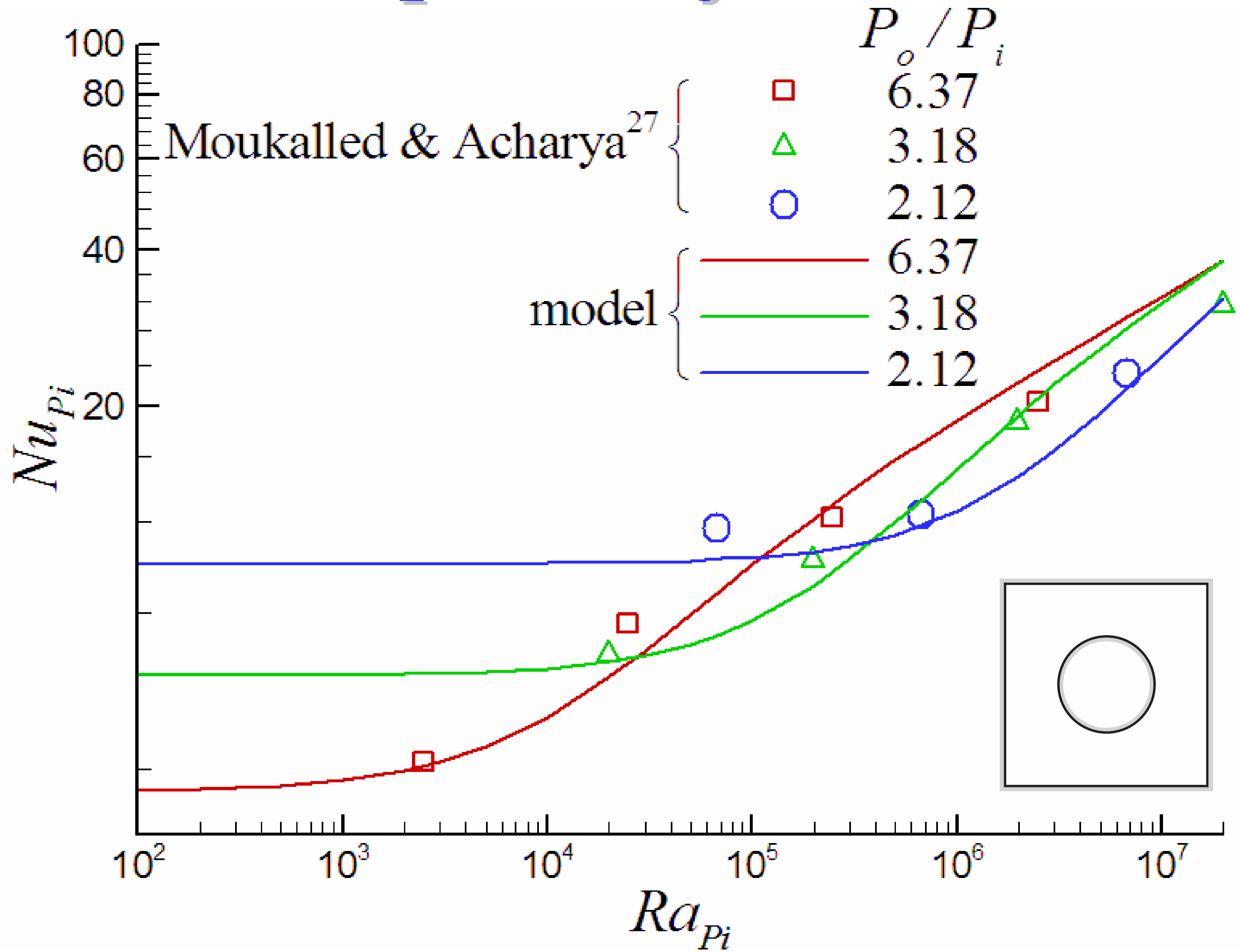
Circle in Square Cylinder



Summary

- Analytical study of natural convection heat transfer for isothermal, horizontal annuli
- Model developed based on combination of analytic, asymptotic relationships
  - Diffusive limit
  - Laminar boundary layer convection
  - Transition flow convection
- Validated using previous data for similar, different inner and outer cylinder shapes
- 6 – 9% RMS difference between model and data