

Modeling Nusselt Numbers for Thermally Developing Laminar Flow in Non-Circular Ducts

Y.S. Muzychka* and M.M. Yovanovich†

Department of Mechanical Engineering
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1

Solutions to thermally developing flow (Graetz Problem) in circular and non-circular ducts are examined. It is shown that the Nusselt number based upon the square root of the cross-sectional flow area is a weak function of the shape of the geometry provided an appropriate aspect ratio is defined. It is also shown that there are two distinct bounds for the fully developed Nusselt number which depend upon the shape and symmetry of the geometry. A general model which is valid for many duct configurations is developed by combining a Leveque model for the thermal entrance region with the fully developed flow asymptote. The new model is simpler than other general models and provides equal or better accuracy. Finally, it is shown that the solution for the elliptic duct geometry may be used to accurately predict results for 8 singly-connected ducts with an accuracy of ± 12 percent.

NOMENCLATURE

A	=	flow area, m^2
a	=	major axis of ellipse or rectangle, m
b	=	minor axis of ellipse or rectangle, m
c	=	linear dimension, m
D	=	diameter of circular duct, m
D_h	=	hydraulic diameter, $\equiv 4A/P$
d^*	=	dimensionless diameter ratio, $\equiv D_h/D_{max}$
$E(\cdot)$	=	complete elliptic integral of the second kind
f	=	friction factor $\equiv \bar{\tau}/(\frac{1}{2}\rho\bar{w}^2)$
$g(\epsilon)$	=	shape function, Eq. 14
h	=	heat transfer coefficient, W/m^2K
k	=	thermal conductivity, W/mK
m	=	area mismatch parameter, $\equiv A_{D_h}/A$
N	=	number of sides of a polygon
n	=	correlation parameter, Eq. (10,12)
$Nu_{\mathcal{L}}$	=	Nusselt number, $\equiv h\mathcal{L}/k$
P	=	perimeter, m
Pr	=	Prandtl number, $\equiv \nu/\alpha$
q	=	heat flux, W/m^2
$Re_{\mathcal{L}}$	=	Reynolds number, $\equiv \bar{w}\mathcal{L}/\nu$
r	=	radius, m
r^*	=	dimensionless radius ratio
s	=	arc length, m
T	=	temperature, K
UWF	=	uniform wall flux
UWT	=	uniform wall temperature
w	=	axial velocity, m/s
\bar{w}	=	average velocity, m/s

x, y, z = cartesian coordinates, m
 $z_{\mathcal{L}}^*$ = dimensionless axial position for
thermally developing flow, $\equiv z/\mathcal{L}Re_{\mathcal{L}}Pr$

Greek Symbols

α = thermal diffusivity, m^2/s
 ϵ = aspect ratio, $\equiv b/a$
 Φ = angular measurement, rad
 γ = symmetry parameter
 μ = dynamic viscosity, Ns/m^2
 ν = kinematic viscosity, m^2/s

Subscripts

\sqrt{A} = based upon the square root of area
 D_h = based upon the hydraulic diameter
 fd = fully developed
 $H, H1$ = based upon isoflux condition
 \mathcal{L} = based upon the arbitrary length \mathcal{L}
 m = mixed or bulk value
 P = based upon perimeter
 T = based upon isothermal condition
 w = wall
 ∞ = fully developed value

Superscripts

o = circular duct limit
 $\overline{(\cdot)}$ = denotes average value of (\cdot)

INTRODUCTION

Thermally fully developed and thermally developing laminar flow heat transfer in circular ducts is discussed in most heat transfer texts^{1,2} and in all convective heat transfer texts³⁻⁵. Results for other common duct configurations encountered in heat transfer problems, namely, the parallel plate, circular duct, rectangular duct, and circular annular duct geometries may also be found in the literature. These geometries have received much attention primarily as a result of their ease to be solved using analytical techniques.

*Graduate Research Assistant

†Professor, Fellow AIAA

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Figures 2 and 3 illustrate many common geometries which have been analyzed for fully developed and thermally developing flow conditions.

Other common and less common geometries have been examined in the fluid mechanics and heat transfer literature. A comprehensive review of this problem was compiled by Shah and London,⁶ while shorter reviews have appeared in Handbooks.⁷ At least forty different geometries have been analyzed using a variety of analytical and numerical techniques for various thermal boundary conditions for both developing and fully developed laminar flow.

This paper has three objectives. The first objective is to present a general approach for accurately predicting the Nusselt number in non-circular ducts for thermally developing (Graetz problem) and fully developed flows, (refer to Fig. 1). Secondly, demonstrate that the square root of the cross-sectional flow area as an alternative to the hydraulic diameter leads to better correlation of the results in non-circular ducts. Finally, develop a simple model which will predict the results for most non-circular ducts for the uniform wall temperature and uniform wall flux conditions.

GOVERNING EQUATIONS AND DIMENSIONLESS GROUPS

The energy equation in cartesian coordinates for thermally developing flow in ducts of constant cross-sectional area is given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{w}{\alpha} \frac{\partial T}{\partial z} \quad (1)$$

where $w = w(x, y)$ is the fully developed velocity profile which may be obtained from the solution to

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{\mu} \frac{dp}{dz} \quad (2)$$

where $\frac{dp}{dz}$ is the pressure gradient which is constant. Equation (2) is subject to the no slip condition $w(x, y) = 0$ at the wall of the duct and to the boundedness $w(x, y) \neq \infty$ condition within the duct cross-section. Yovanovich and Muzychka¹⁷ have developed a simple model which accurately predicts the friction factor in non-circular ducts. It will be shown that this model may be used to accurately predict the Nusselt numbers in non-circular ducts.

Equation (1) is subject to the inlet condition

$$T(x, y, 0) = T_i \quad (3)$$

and the following boundary conditions

$$\begin{cases} q_w = k \frac{\partial T}{\partial n} \Big|_w & \text{UWF} \\ T_w = T_w & \text{UWT} \end{cases} \quad (4)$$

in addition to the boundedness condition, $T(x, y, z) \neq \infty$, at any point within the duct cross-section.

When the flow becomes thermally fully developed the energy equation may be written in terms of the mixing cup temperature⁵ $T_m(z)$,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{w}{\alpha} \frac{dT_m}{dz} \quad (5)$$

for the uniform wall flux (UWF) case, and

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{w}{\alpha} \left(\frac{T_w - T}{T_w - T_m} \right) \frac{dT_m}{dz} \quad (6)$$

for the uniform wall temperature (UWT) case.

The dimensionless heat transfer coefficient or Nusselt number is defined as

$$Nu_{\mathcal{L}} = \frac{\bar{q}_w \mathcal{L}}{k(T_w - T_m)} = \frac{h\mathcal{L}}{k} \quad (7)$$

where T_m is the mixing cup fluid temperature.

In terms of the solutions to Eqs. (1-4), the Nusselt number, $Nu_{\mathcal{L}}$, may be defined as follows

$$Nu_{\mathcal{L}} = \mathcal{L} \frac{\frac{1}{P} \oint \frac{\partial T}{\partial n} \Big|_s ds}{\frac{1}{P} \oint T_w ds - \frac{1}{wA} \iint_A wT dA} \quad (8)$$

where $\frac{\partial T}{\partial n} \Big|_s$ represents the temperature gradient at the duct wall with respect to an inward directed normal, and \mathcal{L} is an arbitrary characteristic length scale to be determined later, A is the cross-sectional area and P is the perimeter. Traditionally, $\mathcal{L} = 4A/P$, the hydraulic diameter of the duct. If the flow is thermally developing, an additional parameter, the dimensionless axial duct position defined as

$$z^* = \frac{z}{\mathcal{L} Re_{\mathcal{L}} Pr} \quad (9)$$

arises in the solution.

Solution to this problem for thermally fully developed flow in a circular duct is discussed in most basic heat transfer texts^{1,2} and all advanced level texts.³⁻⁵ Equations (1-4) may be solved analytically for the circular duct and parallel plate channel, however, the solution requires the evaluation of hypergeometric functions.^{13,18} Sellars, Tribus, and Klein¹⁹ developed approximate mathematical relations for the eigenvalues and eigenfunctions functions for the general solution to the circular duct. Leveque¹¹⁻¹⁴ obtained an asymptotic solution in the entrance region of a circular duct where the thermal boundary layer is thin. A general form of the Leveque solution for non-circular ducts was later proposed by Shah and London⁶ and is discussed in the next section. For non-circular ducts, a full numerical solution to Eqs. (1-4) is required. Numerical solutions have been found for many non-circular ducts and are compiled in Shah and London⁶ and Shah and Bhatti.⁷

PRESENT MODELS

Despite all of the research which has been conducted, one area which has been ignored is the development of simple models which accurately predict the heat transfer in ducts of any cross-sectional shape. Several models have been developed for predicting heat transfer in circular ducts. The earliest of these was a model developed by Hausen⁸ for the Graetz problem in a circular duct. Later, a simple model was proposed by Churchill and Ozoe^{9,10} as part of their development of a more general model for simultaneously developing flow in a circular duct. Churchill and Ozoe^{9,10} combined the solution of Leveque¹¹⁻¹⁴ for the thermal entrance region with the fully developed asymptote.

Presently, only a few studies have been found which approximate the heat transfer in arbitrary shaped ducts. Yilmaz and Cihan^{15,16} developed models to predict the heat transfer characteristics in ducts of arbitrary shape for the Graetz problem. Yilmaz and Cihan^{15,16} developed models for predicting the fully developed Nusselt number for the uniform wall temperature (UWT) and the uniform wall flux (UWF) conditions. They combined these models with a generalized Leveque type solution for the entrance to provide a model which is valid over the entire duct length. These models accurately predict the Nusselt numbers for most duct geometries, however they consist of several equations and are rather cumbersome for engineering calculations, (see Appendix). In this paper a simpler model is proposed which provides equally accurate results for engineering calculations.

MODELING

The development of a model for the Graetz problem will be similar to the development of a model for the hydrodynamic entrance problem.²⁰ A general model for the Graetz problem will be developed by combining a Leveque model for the entrance region with the fully developed flow (see Fig. 1) result using the Churchill-Usagi²¹ asymptotic correlation method.

The proposed model for thermally developing flow takes the form

$$y(z^*) = [y_{z^* \rightarrow 0}^n + y_{z^* \rightarrow \infty}^n]^{1/n} \quad (10)$$

where $y_{z^* \rightarrow 0}$ and $y_{z^* \rightarrow \infty}$ are asymptotic solutions for small and large values of the independent variable z^* and n is the correlation parameter. The method of superposition of asymptotic solutions is discussed in detail by Churchill and Usagi.²¹

In the entrance region where the thermal boundary layer thickness is small, the results are a very weak function of the geometry. A Leveque model for the thermal entrance region may be presented in terms of the friction factor Reynolds number group fRe . For any characteristic length \mathcal{L} this result may be written as

$$Nu_{\mathcal{L}} = C_1 C_2 \left(\frac{fRe_{\mathcal{L}}}{z_{\mathcal{L}}^*} \right)^{\frac{1}{3}} \quad (11)$$

where the constant C_1 determines whether the Nusselt number is an average or local value and the constant C_2 determines whether the boundary condition is UWT or UWF. This asymptotic result is valid in the entrance region all geometries.

Using the Churchill-Usagi²¹ asymptotic correlation method results in the proposed model

$$Nu_{\mathcal{L}}(z^*) = \left(\left\{ C_1 C_2 \left(\frac{fRe_{\mathcal{L}}}{z_{\mathcal{L}}^*} \right)^{\frac{1}{3}} \right\}^n + Nu_{fd}^n \right)^{1/n} \quad (12)$$

Now it is desirable to develop an expression for the fully developed flow asymptote and for the friction factor-Reynolds number group fRe . In a recent paper by the authors¹⁷ a general expression for the fRe group based upon the solution to the elliptic duct was shown to accurately predict the results for other geometries provided that the characteristic length scale $\mathcal{L} = \sqrt{A}$ and a suitable aspect ratio is chosen. This model takes the form

$$fRe_{\sqrt{A}} = 8\sqrt{\pi} \left(\frac{\pi}{4} \frac{(1 + \epsilon^2)}{\sqrt{\epsilon} E(\sqrt{1 - \epsilon^2})} \right) \quad (13)$$

where $E(\cdot)$ is the complete elliptic integral of the second kind, and ϵ is an appropriate aspect ratio of the duct given in Table 1. The aspect ratio ϵ is defined as the ratio of the maximum width and height of each geometry with the constraint that $0 < \epsilon < 1$. Data for many of the geometries shown in Figs. 2 and 3 are shown with the model, Eq. (13), in Fig. 4.

Table 1
Definitions of Aspect Ratio

Geometry	Aspect Ratio
Regular Polygons	$\epsilon = 1$
Singly-Connected [†]	$\epsilon = \frac{b}{a}$
Trapezoid	$\epsilon = \frac{2b}{a+c}$
Annular Sector	$\epsilon = \frac{1-r^*}{(1+r^*)\Phi}$

[†] All except annular sector and trapezoid.

To eliminate the problem of evaluating the elliptic integral, an approximate expression was developed for the shape function $g(\epsilon)$ defined as

$$g(\epsilon) = \left(\frac{\pi}{4} \frac{1 + \epsilon^2}{\sqrt{\epsilon} E(\sqrt{1 - \epsilon^2})} \right) \quad (14)$$

such that

$$fRe_{\sqrt{A}} = 8\sqrt{\pi}g(\epsilon) \quad (15)$$

The shape function $g(\epsilon)$ may be accurately computed from the following approximate expression:

$$g(\epsilon) \approx \left[1.086957^{1-\epsilon}(\sqrt{\epsilon} - \epsilon^{3/2}) + \epsilon \right]^{-1} \quad (16)$$

Equation (16) is valid over the range $0.05 \leq \epsilon \leq 1$ with an RMS error of 0.70 percent and a maximum error less than ± 2 percent which occurs at small values of ϵ .

The general result for the fRe group and the shape function $g(\epsilon)$ will be used next in the development of a general model.

Fully Developed Flow Region

In the fully developed region of the duct, many solutions have been obtained. However, the solutions are strong functions of the duct geometry. It is desirable to have a solution in the fully developed region which is independent of the duct geometry. This may be achieved by selecting a more appropriate characteristic length which will collapse most of the results for non-circular ducts onto a single curve.

In the heat transfer and fluid flow literature, the conventional selection is the hydraulic diameter, $D_h = 4A/P$. In laminar and turbulent flow theory the results for non-circular ducts are usually non-dimensionalized using the hydraulic diameter. This characteristic length arises from a simple control volume force balance performed on an arbitrary slug of fluid in a duct of arbitrary shape²². For a circular duct $D_h = D$, where D is the diameter of the duct. One notable drawback of the hydraulic diameter concept is the fact that the area computed from the hydraulic diameter is not the same as the true area of the duct. This "mismatch" in areas is often assumed to be the cause of the mismatch between the results for the circular duct geometry and non-circular geometries^{4,22}. In the models developed by Yilmaz and Cihan^{15,16} a mismatch parameter which is defined as the ratio of the true duct cross-sectional area to the area computed from the hydraulic diameter is proposed in their development.

The appropriate characteristic length should minimize the differences between solutions for different geometries when the results are non-dimensionalized. Three obvious choices for a characteristic length are the perimeter $\mathcal{L} = P$, the hydraulic diameter $\mathcal{L} = 4A/P$, and the square root of the flow area $\mathcal{L} = \sqrt{A}$. In a recent paper the authors¹⁷ showed by means of dimensional analysis that the square root of the flow area is a more appropriate characteristic length for presenting friction factors of non-circular ducts than the hydraulic diameter. It was shown that most numerical and analytical results for the fully developed friction factor-Reynolds number are predicted to within ± 10 percent by the closed form solution for the elliptical duct when the characteristic length is $\mathcal{L} = \sqrt{A}$.

The approach applied in Yovanovich and Muzychka¹⁷

not only simplifies the results for the fRe number, but may also be used to non-dimensionalize the Nusselt numbers for various flow conditions and thermal boundary conditions. Table 2 compares the Nusselt number for both slug and fully developed flows for both the (UWT) and (UWF) boundary conditions. The Nu_{D_h} results of many polygonal ducts were obtained from Shah and London⁶ and Bejan⁴. These data are from the work of Cheng^{23,24}, Shih²⁵, and Asako et al.²⁶. The results for $Nu_{\sqrt{A}}$ for each flow condition and thermal boundary condition approximately reduce to a single constant for the duct geometries presented. The maximum difference between the triangular duct ($N = 3$) and the circular duct ($N = \infty$) are 16.1 percent and 10.2 percent for the isothermal and isoflux boundary conditions respectively, for the fully developed flow condition. This difference reduces to 8.7 percent and 7.2 percent for the square duct. This difference is much less for the slug flow condition since the uniform velocity distribution includes the corners, whereas for fully developed flow the effect of sharp corners is more pronounced. This exclusion of the corners suggests that an effective flow area which is smaller than the true flow area should be considered. However, for this paper, it will be assumed that the true duct area fully participates. This simplifies the current analysis considerably. The square root of the cross-sectional flow area as a characteristic length is essentially the same as defining an equivalent diameter.

Table 2
Nusselt Numbers for Slug and Fully Developed Flow (FDF) for Regular Polygons

		Isoflux		Isothermal	
	Geometry	FDF [†]	Slug [‡]	FDF [†]	Slug [‡]
Nu_{D_h}	Triangle	3.11	-	2.47	-
	Square	3.61	7.08	2.98	4.93
	Hexagon	4.00	7.53	3.35 [‡]	5.38
	Octagon	4.21	7.69	3.47 [‡]	5.53
	Circular	4.36	7.96	3.66	5.77
$Nu_{\sqrt{A}}$	Triangle	3.51	-	2.79	-
	Square	3.61	7.08	2.98	4.93
	Hexagon	3.74	7.01	3.12	5.01
	Octagon	3.83	7.00	3.16	5.03
	Circular	3.86	7.06	3.24	5.11

[†] Data of Shih (1967) and Cheng (1966, 1969)

[‡] Data of Asako et al. (1988)

The results given in Table 2 are for the polygonal duct geometries. To extend this analysis to geometries which have varying aspect ratios, we will examine the rectangular duct, elliptical duct, and some miscellaneous geometries of elongated shape. Figures 5 and 6 compare the data for the rectangular duct obtained from Shah and London⁶, the elliptical duct obtained from Ebadian, Topakoglu, and Arnas²⁷, and some miscellaneous ducts from Shah and London⁶ when the characteristic length is $\mathcal{L} = \sqrt{A}$. If the results are based upon the square root of cross-sectional

area there is less variation between these geometries for each boundary condition. It is also clear that geometries with angle(s) less than 45 degrees appear to form a lower bound, while geometries with corner angle(s) greater than 45 degrees form an upper bound. Many of the geometries which form the lower bound possess one plane of symmetry with the exception of the rhombus, while all of the geometries which possess more than one plane symmetry form the upper bound.

A model has been developed which accurately predicts the data for the elliptic duct by comparing the solution of the friction factor of Eq. (13) with the data for the Nusselt numbers. The shape function $g(\epsilon)$ which accounts for the aspect ratio effects in the friction factor-Reynolds number group may also be used to obtain a model for the Nusselt number in elliptic ducts. Multiplying the shape function by the Nusselt number for the circular duct gives

$$Nu_{\sqrt{A}} = Nu_{\sqrt{A}}^0 \left(\frac{\pi}{4} \frac{1 + \epsilon^2}{\sqrt{\epsilon} \mathbf{E}(\sqrt{1 - \epsilon^2})} \right) = Nu_{\sqrt{A}}^0 g(\epsilon) \quad (17)$$

where $Nu_{\sqrt{A}}^0$ is equal to 3.24 for the (UWT) boundary condition and 3.86 for the (UWF) boundary condition. This simple expression predicts the data of Ebadian, Topakoglu, and Arnas²⁷ with an RMS error of 3.78 percent for the isothermal boundary condition and 4.70 percent for the isoflux boundary condition.

The Nusselt number for thermally fully developed flow in other non-circular ducts may be approximated by the following relation

$$Nu_{\sqrt{A}} = Nu_{\sqrt{A}}^0 \left(\frac{fRe_{\sqrt{A}}}{8\sqrt{\pi}\epsilon\gamma} \right) \quad (18)$$

The parameter γ is chosen based upon the symmetry and corner angle of the non-circular geometry. Values for γ which define the upper and lower bounds are $\gamma = 1/10$ and $\gamma = -3/10$, respectively. Data for many geometries²⁸⁻³⁶ are shown in Figs. 5 and 6 with the bounds determined by Eq. (18). It is clear that using the square root of the cross-sectional flow area reduces the variation in results of similar geometries. It is also clear that most geometries which do not possess two or more planes of symmetry form a lower bound, while geometries with two or more planes of symmetry form an upper bound. The exception to this rule is the rhombic duct which follows the lower bound due to the small angles formed at vales of $\epsilon < 0.5$.

Entrance Region

Substituting the result from the previous section for the $fRe_{\sqrt{A}}$, a simple Leveque model is obtained for the elliptical duct:

$$Nu_{\sqrt{A}}(z^*) = C_1 C_2 \left(\frac{8\sqrt{\pi}g(\epsilon)}{z^*\sqrt{A}} \right)^{\frac{1}{3}} \quad (19)$$

This new result differs by only 1.2 percent compared to the predictions with Leveque type models derived by

Someswara et al.³⁷ and James³⁸, and 1.5 percent compared to the model derived by Richardson³⁹, for the elliptic duct. The new Leveque model is also much simpler than the models of Someswara et al.³⁷, James³⁸, and Richardson³⁹ which all require numerical integrations. Richardson³⁹ provided a series approximation to avoid numerical integration.

Full Model

Now using the result for the fully developed friction factor presented earlier and the result for the fully developed flow Nusselt number developed in the previous section a new model is proposed having the form

$$Nu_{\sqrt{A}}(z^*) = \left[\left\{ C_1 C_2 \left(\frac{fRe_{\sqrt{A}}}{z^*\sqrt{A}} \right)^{\frac{1}{3}} \right\}^5 + \left\{ C_3 \left(\frac{fRe_{\sqrt{A}}}{8\sqrt{\pi}\epsilon\gamma} \right)^5 \right\}^{\frac{1}{5}} \right] \quad (20)$$

where the constants C_1 , C_2 , C_3 and γ are given in Table 3. These constants define the various cases for local or average Nusselt number and isothermal or isoflux boundary conditions.

Table 3
Constants for Thermally Developing Flow Model in Elliptical Duct

	Local	Average
C_1	1	1.5
	Isothermal (T)	Isoflux (H)
C_2	0.427	0.517
C_3	3.24	3.86
γ	0	0

An optimal value of the correlation parameter n may be obtained for each geometry. In the interest of simplicity it may be chosen to be constant without introducing significant error. Analysis of the available data has shown that the optimal value for the parameter n is $n \approx 5$. The above model accurately predicts the data for all of the geometries examined in this study. The proposed model is considerably simpler than that of Yilmaz and Cihan^{15,16} and is valid for both boundary conditions (UWT or UWF) and for local and average conditions. Comparisons of this new model with data from Shah and London⁶ for the geometries in Figs. 2 and 3 reveal that it predicts the numerical data for many geometries within ± 20 percent. In addition, to its simplicity, the new model is also more flexible, in that both thermal boundary conditions may be handled, whereas the models of Yilmaz and Cihan^{15,16} are different for each thermal boundary condition. In order to provide a model which is more accurate for predicting the results of many geometries, the constants C_2 and C_3 have been modified slightly such that the predicted curve represents

an average value of similar geometries at each value of the aspect ratio $\epsilon = b/a$. These new constants are summarized in Table 4. The accuracy of the model with the modified constants is improved to ± 12 percent. Comparisons of this new model with predictions of the models of Yilmaz and Cihan^{15,16} are given in the next section.

Table 4
Modified Constants for Thermally Developing Flow Models in Non-Circular Ducts

	Local	Average
C_1	1	1.5
	Isothermal (T)	Isoflux (H)
C_2	0.409	0.501
C_3	3.01	3.66
	Upper Bound	Lower Bound
γ	1/10	-3/10

RESULTS

A comparison the proposed model with the available data^{6,40-43} is presented in Figs. 7-9. Table 5 presents a summary of the maximum and minimum percent differences between the data and the proposed model. A comparison of the proposed model with the models of Yilmaz

and Cihan^{15,16} is also presented in Table 5. Good agreement between the model and data is observed for all of the geometries except the isosceles triangular duct for the UWT boundary condition. In the case of the elliptic duct, no published data are available for comparison. However, Yilmaz and Cihan^{15,16} provide comparisons of their model with their own numerical data. A comparison of the proposed model with that of Yilmaz and Cihan^{15,16} shows that good agreement should be obtained if direct comparison with their data were possible. Also, the proposed model is developed from asymptotic solutions for the elliptic duct. Thus the model is expected to provide very accurate results for the elliptic duct geometry. This particular geometry is extremely important in heat exchanger design where large heat transfer coefficients are desired. A comparison with the data for the parallel plate channel is also provided. For this geometry $\sqrt{A} \rightarrow \infty$, however, this geometry is accurately approximated by the rectangular duct when $\epsilon = 0.01$. Good agreement is obtained with the current model when the parallel plate channel is modeled as a finite area duct with low aspect ratio. In all cases the proposed model provides equal or better accuracy than the models of Yilmaz and Cihan^{15,16} and is also much simpler. Finally, the proposed model is able to determine local or average Nusselt numbers, whereas the models of Yilmaz and Cihan^{15,16} were developed for the average Nusselt number (UWT) and local Nusselt number (UWF).

Table 5
Comparison of Percent Differences* Between Models and Data for Thermally Developing Flow

Ref.	Geometry	$Nu_{\sqrt{A}}^T(z^*)$ (Average)		$Nu_{\sqrt{A}}^H(z^*)$ (Local)	
		Proposed Model	Yilmaz and Cihan (1993)	Proposed Model	Yilmaz and Cihan (1995)
6	Circle	-1.24/8.57	-0.47/-3.89	-1.60/5.27	-3.83/0.70
40	Rectangle $\epsilon = 1$	-7.53/-1.07	-3.52/0.58	-2.76/1.73	-3.95/4.80
41	Rectangle $\epsilon = 0.5$	-4.26/1.38	-0.86/4.16	-3.16/1.31	-4.1/0.24
41	Rectangle $\epsilon = 0.25$	3.01/10.85	0.54/7.43	1.80/7.01	0.44/8.20
41	Rectangle $\epsilon = 0.167$	3.84/11.86	3.32/1.59	-	-
6	Rectangle $\epsilon \rightarrow 0$	1.6/10.0	0.24/3.70	-9.93/7.03	-2.2/5.86
41,42	Isosceles Triangle $2\phi = 30^\circ$	25.75/3.92	33.8/0.88	-	-
41,42	Isosceles Triangle $2\phi = 60^\circ$	-12.11/-6.81	-4.36/-1.29	-7.24/-2.09	-4.79/-1.38
41,42	Isosceles Triangle $2\phi = 90^\circ$	-24.45/1.56	-12.28/-1.97	-9.88/3.35	-3.72/0.76
43	Semi-Circle $\epsilon = 0.5$	-	-	-7.90/7.78	-0.91/8.83
15,16	† Ellipse $\epsilon = 0.9$	5.61/11.81	-	3.46/7.85	-4.7/-1.2
15,16	† Ellipse $\epsilon = 0.8$	3.35/10.82	-2.66/-1.06	1.14/6.77	-5.8/-0.8
15,16	† Ellipse $\epsilon = 0.6$	-5.82/7.18	1.4/4.7	-8.23/4.04	-

* %diff= (Analytical - Predicted)/(Analytical) $\times 100$

† Comparison of proposed model with results predicted by model of Yilmaz and Cihan^{15,16}

SUMMARY

A simple model for predicting the Nusselt numbers for fully developed flow and thermally developing flow conditions was developed for both the uniform wall temperature and uniform wall flux boundary conditions. The proposed model only requires three parameters, the aspect ratio of the duct, the dimensionless duct length, and a geometry parameter, whereas the models of Yilmaz and Cihan^{15,16} consist of several parameters and equations. The model predicts most of the thermally developing flow data available in the literature to within ± 12 percent for 8 singly connected ducts. The proposed model may also be used to accurately predict results for ducts which no solution or tabulated data exist. Finally, it was shown that the square root of the cross-sectional flow area was more effective than the hydraulic diameter at collapsing the results of geometries having similar shape and aspect ratio.

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APPENDIX

Yilmaz and Cihan^{15,16} developed models for the Graetz problem in a duct of arbitrary shape. Each of these models follows a similar development, however, the model for the constant temperature or isothermal boundary condition is for the average Nusselt number, whereas the model for the constant flux or isoflux boundary condition is for the local Nusselt number. No explanation is given for this development. The basic equations for each model are presented below in Table 6. In both cases $m = A/A_{D_h}$ and $d^* = D_h/D_{max}$ where A_{D_h} is the area based upon the hydraulic diameter, $A_{D_h} = \pi D_h^2/4$, and D_{max} is the diameter of the maximum inscribed circle.

Table 6
Models of Yilmaz and Cihan^{15,16}

Isothermal (UWT)	Isoflux (UWF)
$Nu_m = Nu_\infty \left\{ 1 + 4.212 \frac{\Psi \Phi^3}{z^* Nu_\infty} - 0.8 \left(\frac{\Psi \Phi^3}{z^* Nu_\infty} \right)^{2/3} \right\}^{1/3}$	$Nu_z = Nu_\infty \left\{ 1 + 2.212 \frac{\Psi \Phi^3}{z^* Nu_\infty} - 0.76 \left(\frac{\Psi \Phi^3}{z^* Nu_\infty} \right)^{2/3} \right\}^{1/3}$
$\Phi = 1 + \frac{3(d^*/2)^{7/8} - 1}{1 + \frac{0.25}{m-1}}$	$\Phi = 1 + \frac{3(d^*/2)^{2/3} - 1}{1 + \frac{0.4}{m-1}}$
$\Psi = 1 + \frac{\psi_\infty - 1}{1 + 0.33 \frac{d^{*2.25}}{m-1}}$	$\Psi = 1 + \frac{\psi_\infty - 1}{1 + 0.33 \frac{d^{*2.25}}{m-1}}$
$\psi_\infty = \frac{3}{8} d^{*2} (3 - d^*)$	$\psi_\infty = \frac{3}{8} d^{*2} (3 - d^*)$
$Nu_\infty = 3.657 \left(1 + \frac{\phi_\infty - 1}{1 + \frac{1}{m-1}} + \Delta\phi \right)$	$Nu_\infty = 4.364 \left(1 + \frac{\phi_\infty - 1}{1 + \frac{0.6}{m-1}} \right)$
$\phi_\infty = 1.4153 \frac{7/3 - d^*}{(3/d^* - 1)^2}$	$\phi_\infty = 1.4153 \frac{7/3 - d^*}{(3/d^* - 1)^2}$
$\Delta\phi = \Delta\phi_{max} \frac{0.95(m-1)^{0.5}}{1 + 0.038(m-1)^3}$	-
$\Delta\phi_{max} = \frac{7 \times 10^{-3} d^{*8}}{(1 + 10d^{*-28})(1 + 64 \times 10^{-8} d^{*28})^{0.5}}$	-

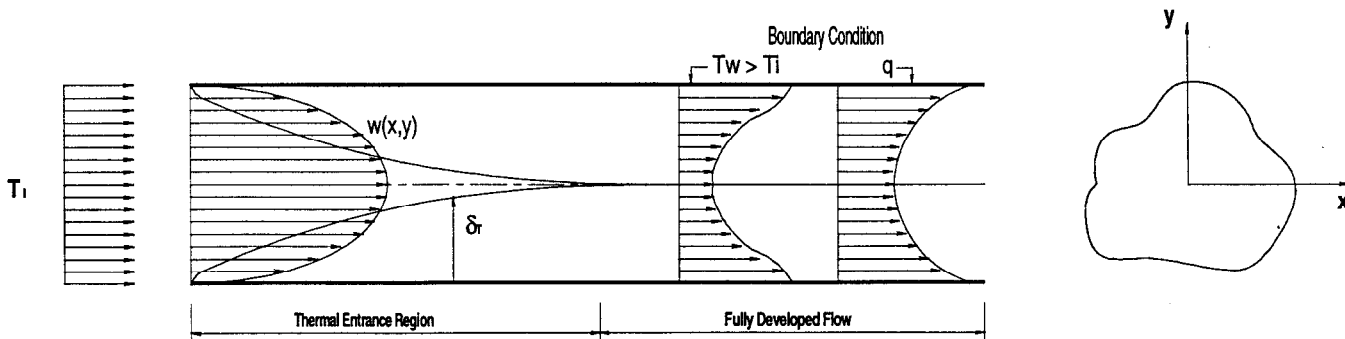


Fig. 1 - Thermal entrance problem.

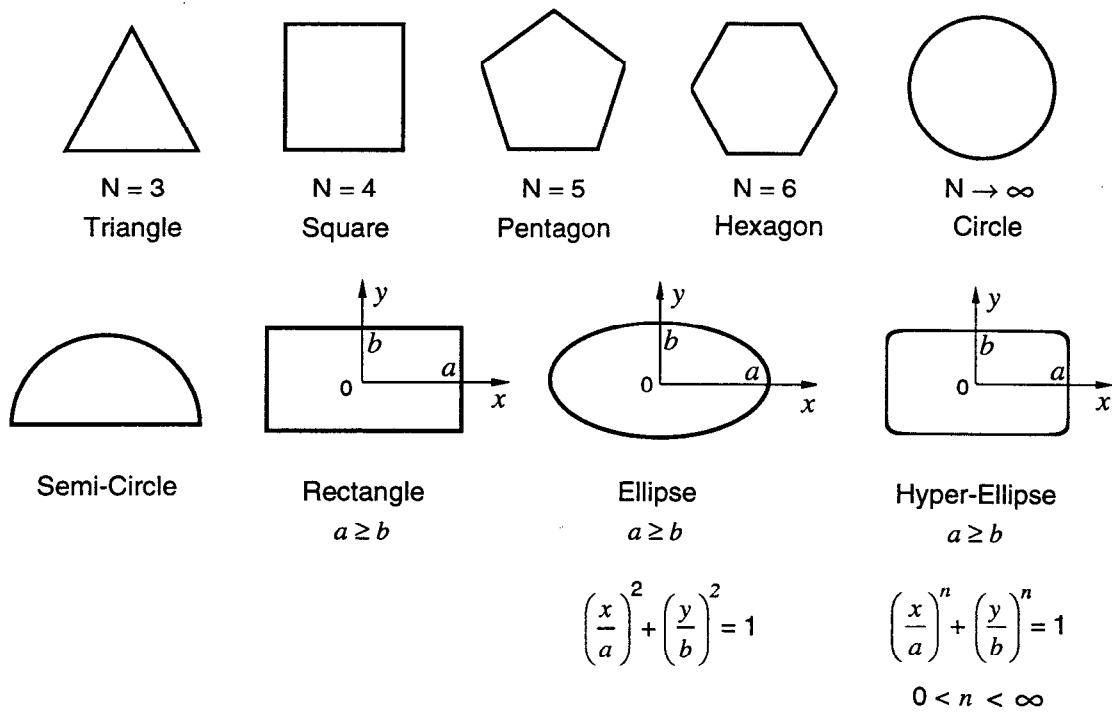


Fig. 2 Common singly-connected geometries

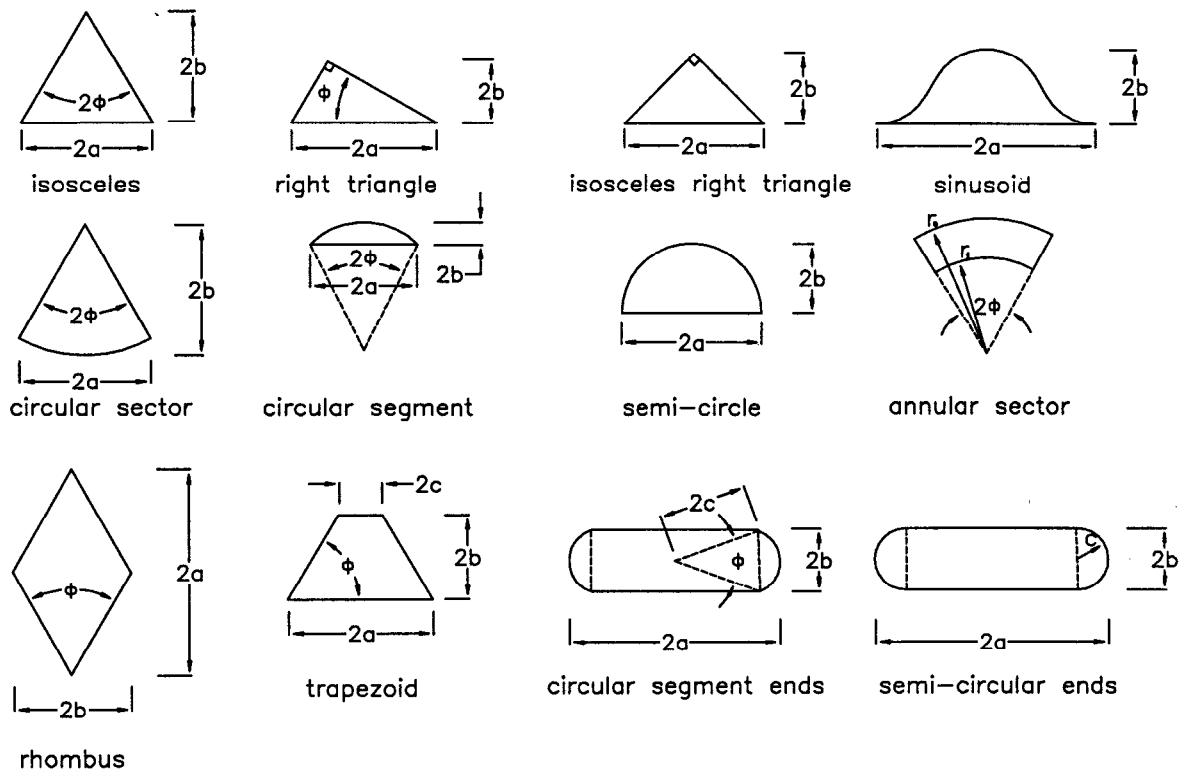


Fig. 3 Other singly-connected geometries

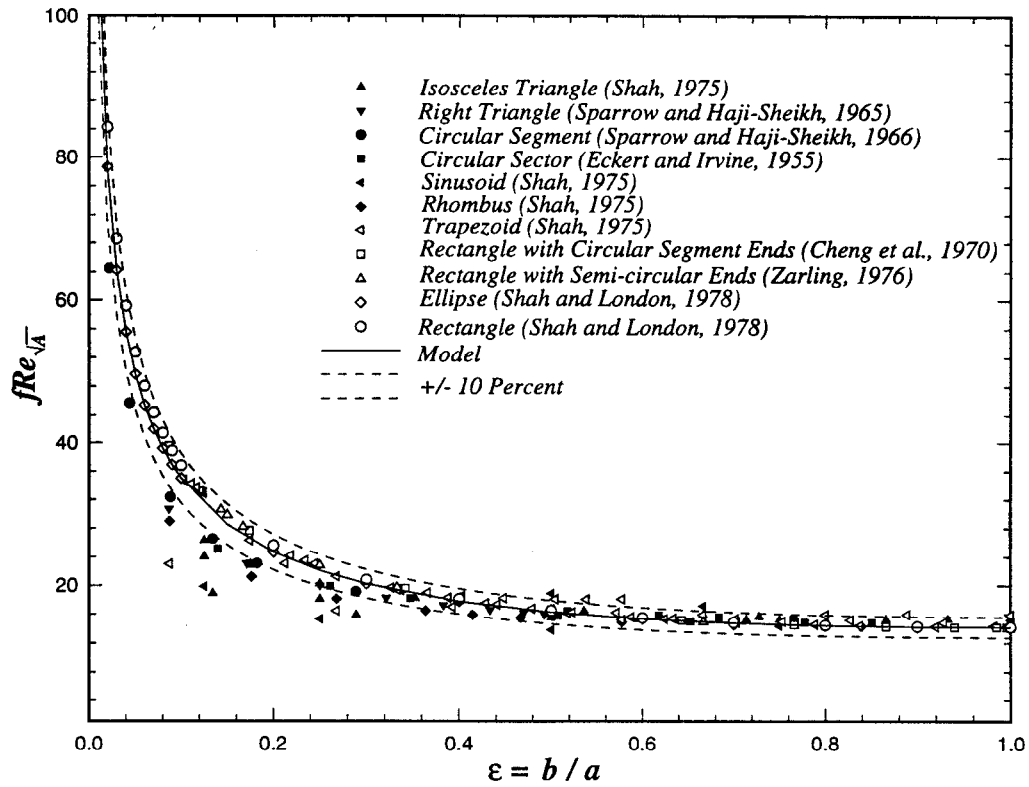


Fig. 4 Fully developed flow $fRe\sqrt{A}$

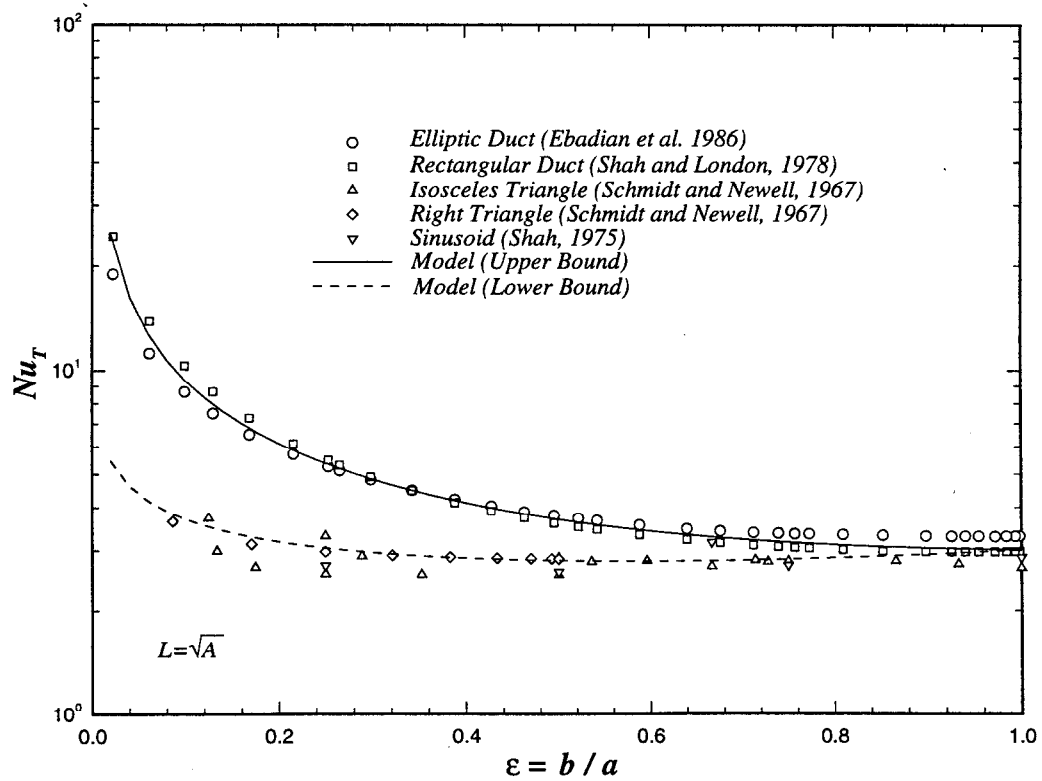


Fig. 5 Fully developed flow Nu_T

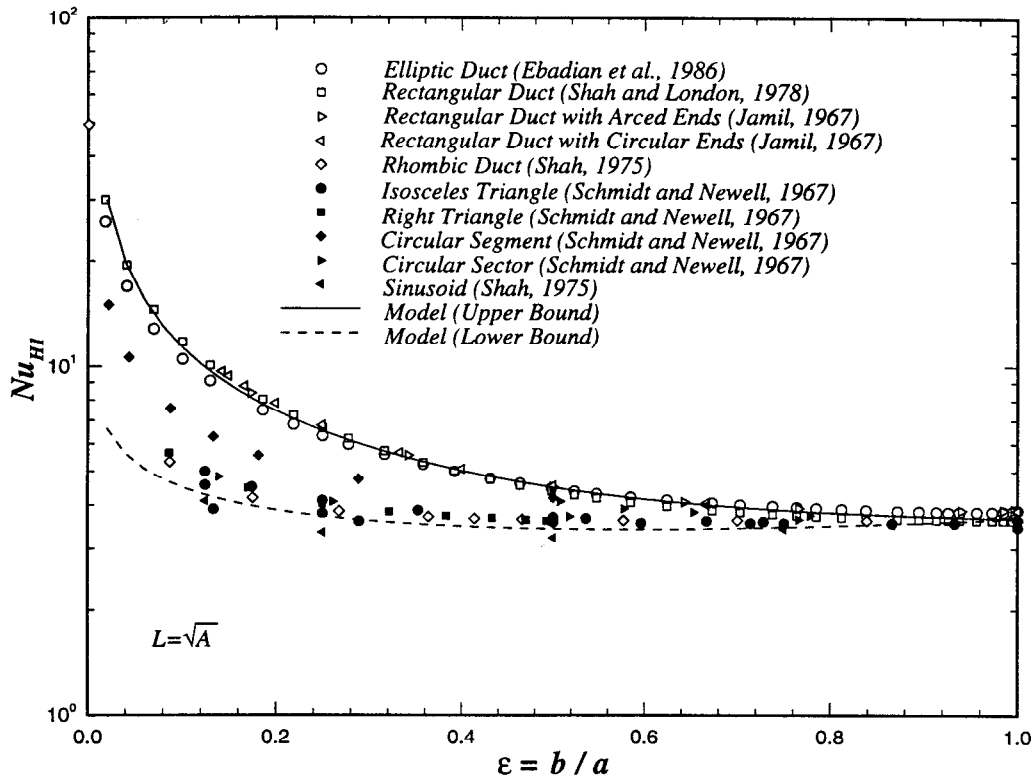


Fig. 6 Fully developed flow Nu_H

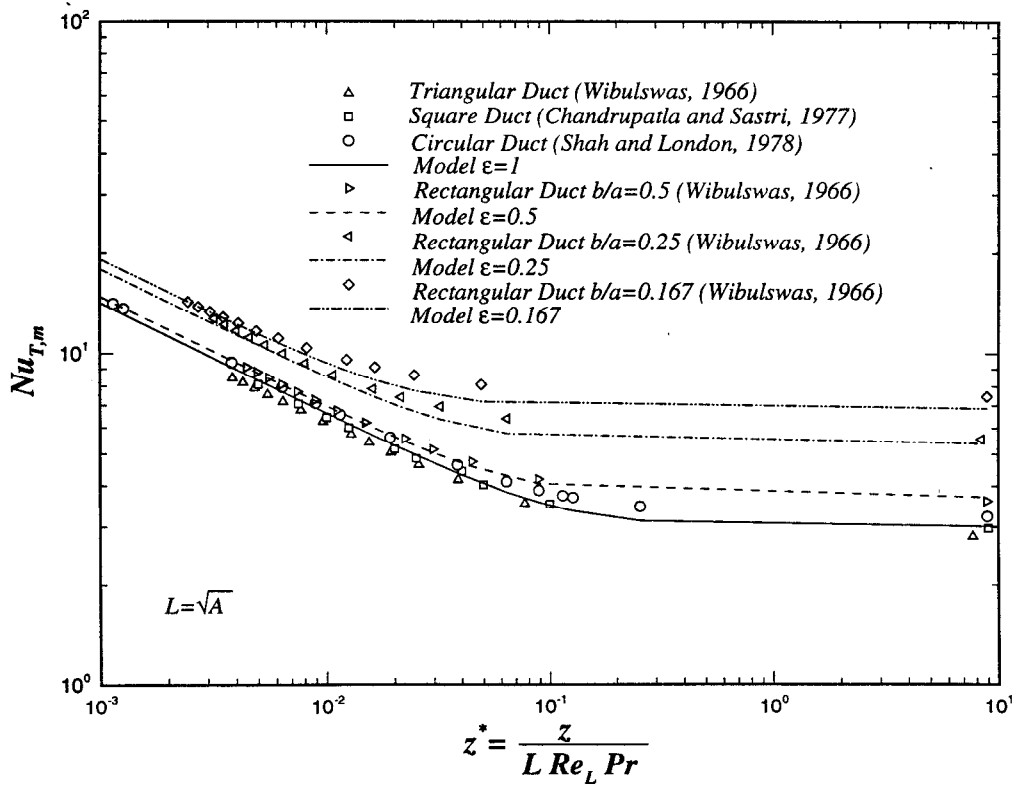


Fig. 7 Thermally developing flow $Nu_{T,m}$

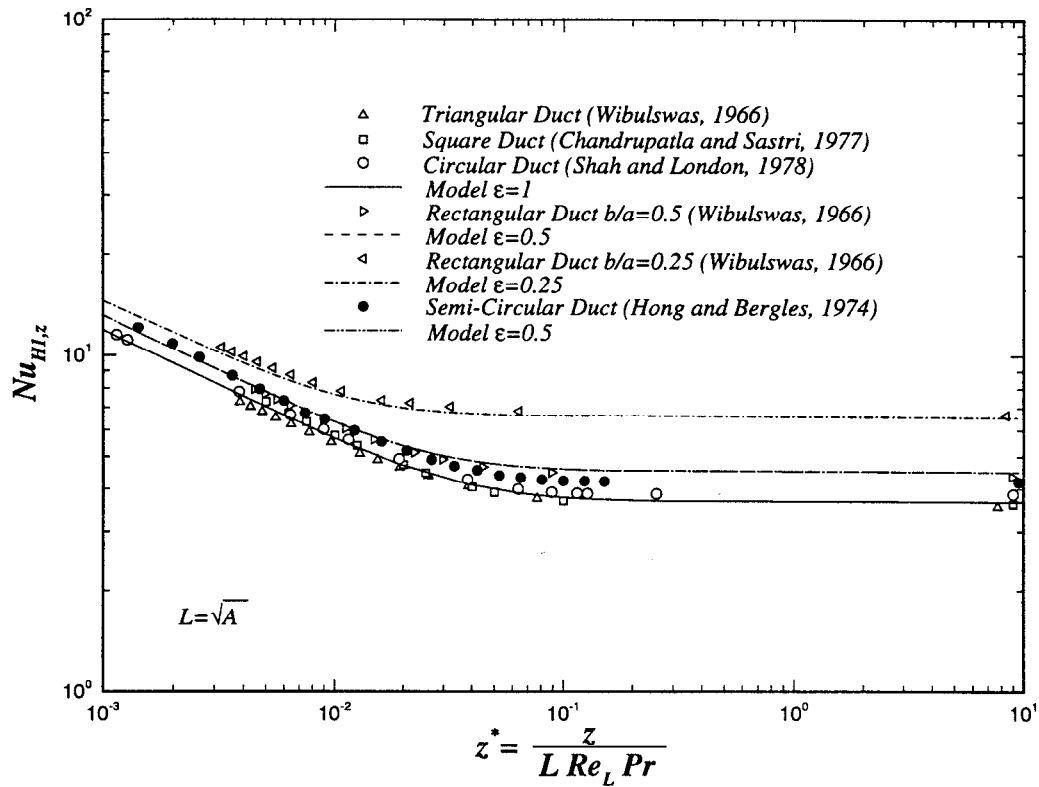


Fig. 8 Thermally developing flow $Nu_{H,z}$

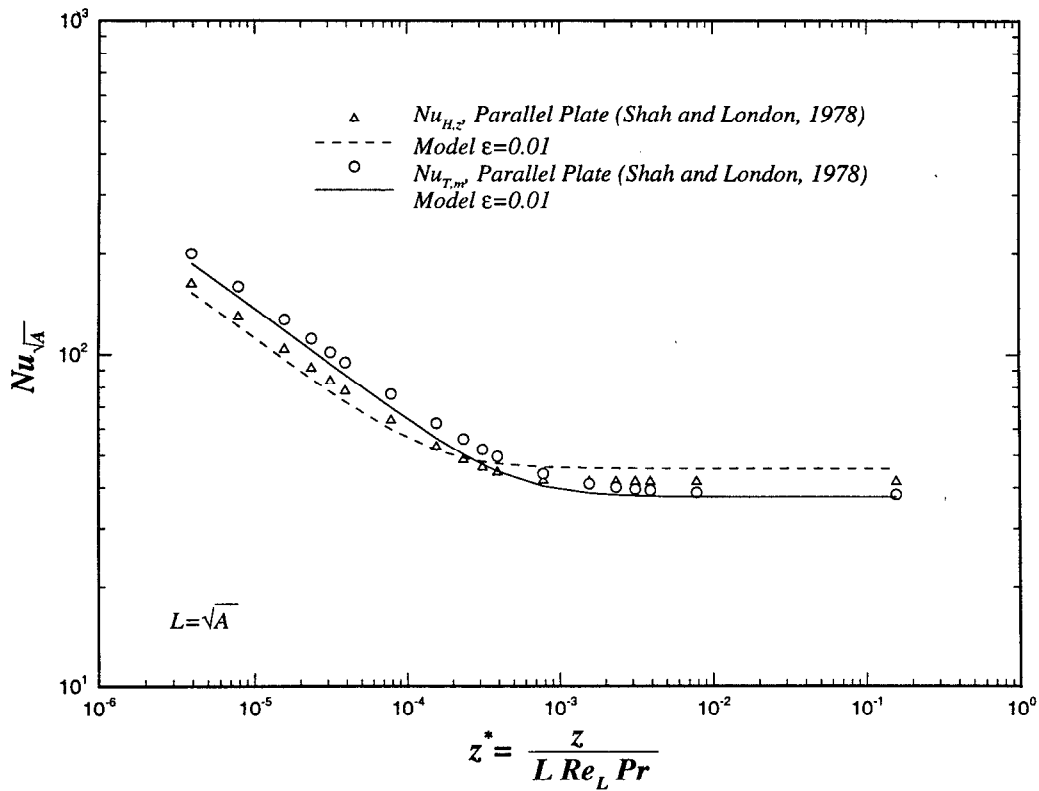


Fig. 9 Thermally developing flow (Parallel Plates)