Analytical Modeling of Spreading Resistance in Flux Tubes, Half Spaces, and Compound Disks

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***Abstract*—A review of previously published models and solutions pertinent to the issue of modeling thermal resistances of diamond on copper heat sink systems is presented. The many particular solutions are shown to be special cases of the comprehensive model developed for a circular heat source in perfect thermal contact with the top surface of a compound disk which consists of two isotropic layers in perfect thermal contact. The bottom surface of the compound disk is subjected to a convective or contact cooling condition. Whenever possible simple models and correlation equations are presented for ease of computation. Bounds are presented for estimating the overall thermal resistance of several important cases.**

***Index Terms*—Compound disks, flux tubes, half space, spreading and constriction resistance.**

NOMENCLATURE

a	Source radius (m).
A_c	Contact area (m^2).
A_t	Flux tube area (m^2).
b	Heat spreader radius (m).
Bi	Biot number $\equiv hb/k_1$.
$E(\cdot)$	Complete elliptic integral.
h	Convective coefficient ($\mathrm{W/m^2K}$).
$J_0(\cdot), J_1(\cdot)$	Bessel functions.
k	Thermal conductivity (W/mK).
q	Heat flux ($\mathrm{W/m^2}$).
Q	Heat flow rate (W).
R	Thermal resistance (°C/W).
t	Layer thickness (m).
T	Temperature (°C).
r, z	Polar coordinates.

Greek Symbols

δ_n	Eigenvalues.
ϵ	Relative source size $\equiv a/b$.
κ	Conductivity ratio $\equiv k_1/k_2$.
μ	Heat flux distribution parameter (see Fig. 2).
τ	Dimensionless thickness $\equiv t/b$.
ϕn	Subfunction (16).
φn	Subfunction (17).
ψ	Dimensionless spreading resistance.

Subscripts

1, 2	First and second layers.
c	Constriction or spreading.

Superscripts

q	Isoflux solution.
T	Isothermal solution.

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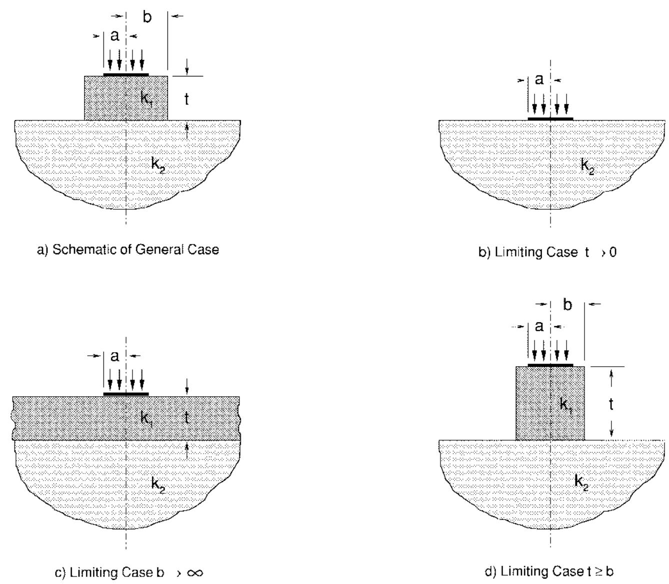

Fig. 1. Typical spreading resistance problems.

I. INTRODUCTION

THE recently published paper of Hui and Tan [1] was the motivation for this review article. In their paper, an elegant mathematical solution was presented for the general problem depicted in Fig. 1(a) which shows a circular heat source of radius a in perfect thermal contact with a heat spreader modeled as a circular disk of radius b, thickness t and thermal conductivity k_1 which is in perfect thermal contact with a half-space of thermal conductivity k_2. They assumed a uniform heat flux distribution over the heat source area, and all other free boundaries were taken to be adiabatic. They also considered the special cases shown in Fig. 1(b) and (c), where the radius of the spreader $b \to \infty$ and they presented the second solution as a semi-infinite integral. Solutions for the

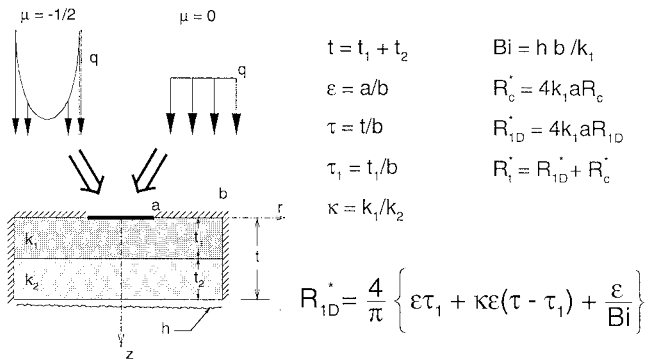

Fig. 2. General two layer compound disk problem.

centroid and area-average temperatures of the heat source were given. They presented numerical results only for $k_1/k_2 = 4$ corresponding to a diamond-copper system, for several values of the relative spreader thickness: $0.25 \le t/a \le 3.0$, and for several values of the relative spreader size: $3 \le b/a < \infty$.

Hui and Tan [1] did not present simplifications of their solutions; they did not indicate whether their solution could be used to handle the problem depicted in Fig. 1(d); and their review of other pertinent publications was limited.

One objective of this review is to bring to the readers attention the numerous related publications that give solutions to particular problems that are not handled by the results of [1]. A second objective is to present simplifications, correlation equations and approximations that have been presented by several researchers. A third objective is to present the comprehensive solution developed by [2] for the system depicted in Fig. 2 which shows a circular heat source of radius a in perfect contact with the top surface of a compound circular disk of radius b and overall thickness t which is cooled over its entire bottom surface through either a uniform convective or contact conductance h. The compound disk consists of two isotropic materials of thermal conductivities: k_1, k_2, and thicknesses: t_1, t_2, respectively. The free surfaces of the compound disk are adiabatic and the heat flux over the heat source region is either uniform or has the shapes shown in Fig. 2. It will be shown that the general solution presented in this paper contains the particular results presented by [1], and several important results which appear in other published works.

II. General Review of Flux Tube and Half-Space Solutions

The solutions pertinent to this topic have been obtained for the heat flux tube and the half-space (or semi-infinite space). Since the flux tube solutions are general, they will reduce to the half-space solutions.

The flux tube solutions are given in terms of infinite series, whereas the half-space solutions are given in terms of integrals.

A. Flux Tube Solutions

The general review begins with the flux tube solutions. The flux tube consists of a circular heat source area of radius a which is in perfect thermal contact with a layer of radius b and thickness t_1, as shown in Fig. 3(c). The layer is in perfect contact with the substrate whose thermal conductivity is k_2 and with thickness $t_2 \to \infty$. The thermal conductivities are assumed to be isotropic. The thermal spreading (constriction) resistance has been obtained for the isoflux and isothermal boundary conditions specified over the heat source area. Other boundary conditions have also been examined. The solutions have been reported for

1) conductive layers, $k_1 > k_2$;
2) resistive layers, $k_1 < k_2$.

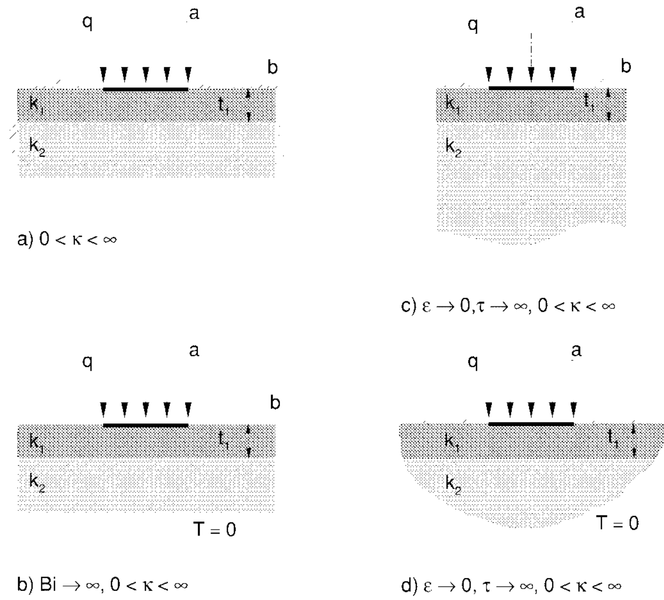

Fig. 3. General two layer compound disk and three special cases.

Antonetti and Yovanovich [3] presented an analytical solution for a single, conductive layer for both isothermal and isoflux conditions. Board [4] has provided analytical solutions for the effect of multiple layers on the spreading resistance. Hui and Tan [1] presented analytical solutions for conductive layers for the isoflux boundary condition. Kennedy [5] presented several analytical solutions for $k_1 = k_2$ for the maximum and area-average source area temperature for the isoflux boundary condition. Mal'kov *et al.* [6] examined the effect of soft metal coatings and linings on the spreading resistance. Mikic and Carnasciali [7] presented an approximate solution for determining the effect of thermal conductivity of plating materials on the spreading resistance. In a recent paper [8] presented analytical solutions for determining the effects of relative thicknesses and relative thermal conductivities of multiple layers. They examined the isoflux and the equivalent isothermal flux boundary conditions. Their solutions are valid for any combination of thermal conductivities. Negus and Yovanovich [9] presented accurate correlation equations of the dimensionless spreading resistance for the isoflux and equivalent isothermal flux boundary conditions for the case of $k_1 = k_2$. In a companion paper [10], the method of optimized images was used to calculate accurately the spreading resistance for the isothermal boundary condition for $k_1 = k_2$. Negus *et al.* [11] examined the effect of boundary conditions on the thermal constriction (spreading) resistance of a single conductive or resistive layer. Negus *et al.* [12] demonstrated for the isoflux boundary condition that the dimensionless spreading resistance defined as $k_1\sqrt{A_c}R_c$ is a weak function of the relative size of the heat source area defined as $\sqrt{A_c/A_t}$ where A_c is the source area and A_t is the flux tube cross section area. They examined three combinations:

1) circular area on circular flux tube;
2) square area on a square flux tube;
3) circular area on a square flux tube.

The results were obtained for $k_1/k_2 = 1$. Schankula *et al.* [13] presented analytical and experimental results for the effect of oxide films on the constriction resistance of zirconium alloys for nuclear applications. Simon *et al.* [14] presented analytical results for the analogous problem of current flow from an isopotential circular source area into a circular flux tube. Yip [15] reported analytical and experimental results for the effect of oxide films on thermal constriction resistance. Yovanovich [16] developed a general solution for arbitrary axisymmetric flux distributions for $k_1 = k_2$. By means of the general solution he then presented the general solution for a family of axisymmetric flux distributions of the form: $C(1-u^2)^{\mu}$ where C is a constant, $u = r/a$ is any point in the source area, and μ is a flux distribution parameter. Yovanovich presented three solutions for $\mu = -1/2$, 0, 1/2. He also reported numerical values for the dimensionless constriction (spreading) resistance for a range of relative source size $\epsilon = a/b$ as well as some correlation equations.

B. Half-Space Solutions

Several studies have produced results for the circular heat source area of radius a placed in perfect thermal contact with an isotropic layer of thermal conductivity k_1 which is in perfect thermal contact with an isotropic half-space of thermal conductivity k_2, as shown in Fig. 3(d). The dimensionless spreading resistance in this case depends on the relative layer thickness t_1/a, the relative layer thermal conductivity k_1/k_2, and the boundary condition over the source area.

Beck *et al.* [17] presented a novel surface element method for calculating the maximum temperature. They developed a set of convenient algebraic equations for calculation of the maximum temperature for a diamond layer on a copper half-space for the isoflux boundary condition. Board [18] presented the solution for the isoflux annular source on a single layer in contact with a half-space. He developed simple approximate expressions for both conductive and resistive layers for the circular source. Dryden [19] developed an analytical solution for the equivalent isothermal boundary condition for a single layer. He presented approximate solutions valid for thin and thick layers for both conductive and resistive layers. In a second paper Dryden *et al.* [20] developed short and long time solutions for the effect of a single layer which is either conductive or resistive, and for arbitrary, axisymmetric flux distributions. Hui and Tan [1] also developed the solution for an isoflux source on a single layer. Yovanovich [21] developed a surface element method for determining the constriction (spreading) resistance of arbitrary singly or doubly-connected heat source areas which are subjected to the isoflux boundary condition for the case where $k_1 = k_2$.

C. Finite Circular Disk Solutions

Solutions are presented for calculating the spreading resistance from a circular source of radius a which is in perfect contact with a circular disk of radius b. The circular disk consists of two isotropic layers in perfect contact, the first layer adjacent to the source has a thickness t_1 and a thermal conductivity k_1 and the second layer has a thickness t_2 and thermal conductivity k_2, as shown in Fig. 3(a). The lower face of the disk is in contact with a thermal sink through a uniform convective or contact conductance h. The free surfaces of the disk are adiabatic.

The dimensionless constriction (spreading) resistance will be a function of the boundary condition over the heat source area, the basis for the constriction resistance (average or maximum source temperature), the relative layer thicknesses: $t_1/a, t_2/a$, the relative conductivity k_1/k_2, the relative size of the heat source $\epsilon = a/b$ and the boundary condition at the sink boundary $Bi = hb/k$. The solution to this general problem clearly contains the solutions described above. Kennedy [5] presented the solution for the maximum temperature for the isoflux source for $k_1/k_2 = 1$ and $Bi = \infty$. In a technical note [22], the analytical solution for the isoflux circular source was presented. They also proposed an approximate relationship for the ratio of the spreading resistance with a layer to the spreading resistance without a layer. The simple relationship is reported to be accurate to approximately 30%. Yovanovich *et al.* [2] presented the most comprehensive solution valid for any axisymmetric flux distribution over the source area. They reported analytical results for three flux distributions. Saabas *et al.* [23] developed the analytical solution for the isoflux circular source area and the isoflux annular area placed in perfect contact with a compound disk. The solution can handle the special case of a circular heat source and a circular heat sink with uniform flux over both areas. Nelson and Sayers [24] reported in tabular and graphical form the results of an extensive numerical study for the isoflux source. In two related papers [25], [26], analytical solutions for the isoflux circular source were presented. They reported expressions for the area-average and maximum temperatures. They also proposed a simple closed form expression which they reported is accurate to within 10% of the full solution. They reported that their computed full solution results were in excellent agreement with the numerical values reported by [24].

Since the solution for the compound disk is more general than the flux tube and half-space solutions, it will be considered in the subsequent section. The general solution of [2] will be examined in detail to reveal its characteristics and to show that it reduces to the particular solutions presented in the papers reviewed above.

III. Spreading Resistance Within Compound Disks

The compound disk is shown in Fig. 2. The disk consists of two isotropic materials of thickness: t_1, t_2 and thermal conductivities: k_1, k_2 which are in perfect contact. The radius of the compound disk is denoted b and its thickness is denoted $t = t_1 + t_2$. The lateral boundary $r = b$ is adiabatic, the face at $z = t$ is either cooled by a fluid through the film

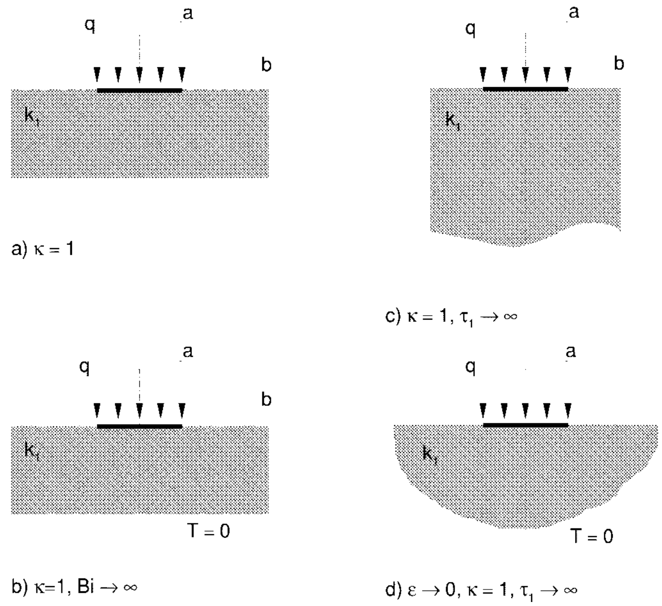

Fig. 4. Special cases of the two layer compound disk for $k_2 = k_1$.

conductance h or it is in contact with a heat sink through a contact conductance h. In either case h is assumed to be uniform. The face at $z = 0$ consists of the heat source area of radius a and the remainder of that face $a < r \le b$ is adiabatic. The boundary condition over the source area can be modeled as

1) uniform heat flux;
2) isothermal.

The complete solution for these two boundary conditions has been given by [2]. The general solution for the dimensionless spreading parameter $\psi = 4k_1 a R_c$ depends on several dimensionless parameters: $\tau = t/b, \tau_1 = t_1/b, \tau_2 = t_2/b, \epsilon = a/b, \kappa = k_1/k_2, Bi = hb/k_2, \mu$. The parameter μ defines the heat flux distribution over the contact area. When $\mu = 0$, the heat flux is uniform (isoflux), and when $\mu = -1/2$, this heat flux distribution is called the equivalent isothermal distribution because it produces an *almost* isothermal contact area provided $a/b <$ 0.6. The general compound disk solution given below reduces to the several special cases shown previously in Figs. 3 and 4.

A. Mathematical Formulation

The governing equation for the steady-state axisymmetric temperature distributions within the layer $0 \le z \le t_1$ of thermal conductivity k_1 and within the substrate $t_1 \le z \le t = t_1 + t_2$ of thermal thermal conductivity k_2 is

$$\nabla^2 T_i = 0, \qquad i = 1, 2 \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{2}$$

The boundary condition along the axis $r = 0$ in both regions is the symmetry condition:

$$\frac{\partial T_i}{\partial r} = 0, \qquad i = 1, 2. \tag{3}$$

The boundary condition along the lateral boundary $r = b$ in both regions is the adiabatic condition

$$\frac{\partial T_i}{\partial r} = 0, \qquad i = 1, 2. \tag{4}$$

The boundary conditions over the top surface $z = 0$ of the first layer are

$$-k_1 \frac{\partial T_1}{\partial z} = q(r), 0 \le r < a \quad \text{and} \quad -k_1 \frac{\partial T_1}{\partial z} = 0, a < r \le b \tag{5}$$

where the heat flux distribution over the heat source area $0 \le r \le a$ can be

1) uniform where $q(r) = Q/(\pi a^2)$;
2) the equivalent isothermal heat flux distribution $q(r) = Q/2\pi a\sqrt{a^2 - r^2}$ where Q is the total heat transfer rate dissipated by the heat source.

The perfect contact boundary conditions along $z = t_1, 0 \le r \le b$ are

$$T_1 = T_2, \qquad -k_1 \frac{\partial T_1}{\partial z} = -k_2 \frac{T_2}{\partial z} \tag{6}$$

The final boundary condition along the lower face $z = t$, $0 \le r \le b$ is the Robin condition

$$-k_2 \frac{\partial T_2}{\partial z} = h(T_2(r,t) - T_{\text{ref}}) \tag{7}$$

where T_{ref} is some convenient reference temperature.

B. Components of Total Thermal Resistance

The total resistance of the system is defined as

$$QR_{\text{total}} = \overline{T}_1(0) - T_{\text{ref}} \tag{8}$$

where the area-mean source temperature is defined as

$$\overline{T}_1(0) = \frac{1}{\pi a^2} \int_0^a T_1(r, 0) 2\pi r \, dr. \tag{9}$$

The total thermal resistance can be written in terms of two component resistances

$$R_{\text{total}} = R_c + R_{1D} \tag{10}$$

where the one-dimensional (1-D) conduction resistance of the system is

$$R_{1D} = \frac{t_1}{k_1 \pi b^2} + \frac{t_2}{k_2 \pi b^2} + \frac{1}{h \pi b^2} \tag{11}$$

and the spreading (constriction) resistance is denoted as R_c. It is convenient to write the spreading resistance in its dimensionless form, where the general solution for dimensionless spreading resistance is [2]

$$\psi = \frac{8(\mu + 1)}{\pi \epsilon} \sum_{n=1}^{\infty} A_n(n, \epsilon) B_n(n, \tau, \tau_1) \frac{J_1(\delta_n \epsilon)}{\delta_n \epsilon}. \tag{12}$$

The coefficients A_n depend on the heat flux parameter μ. They become for $\mu = -1/2$

$$A_n = \frac{-2\epsilon \, \sin(\delta_n \epsilon)}{\delta_n^2 J_0^2(\delta_n)} \tag{13}$$

and for $\mu = 0$

$$A_n = \frac{-2\epsilon J_1(\delta_n \epsilon)}{\delta_n^2 J_0^2(\delta_n)}. \tag{14}$$

The function B_n is defined as

$$B_n = \frac{\phi_n \tanh(\delta_n \tau_1) - \varphi_n}{1 - \phi_n} \tag{15}$$

and the two functions which appear in the above relationship are defined as

$$\phi_n = \frac{\kappa - 1}{\kappa} \cosh(\delta_n \tau_1)[\cosh(\delta_n \tau_1) - \varphi_n \sinh(\delta_n \tau_1)] \tag{16}$$

and

$$\varphi_n = \frac{\delta_n + Bi \tanh(\delta_n \tau)}{\delta_n \tanh(\delta_n \tau) + Bi}. \tag{17}$$

The eigenvalues δ_n are the positive roots of $J_1(\delta_n) = 0$. They can be computed quickly and accurately by means of the modified Stokes approximation [8]

$$\delta_n = \frac{\beta_n}{4}\left[1 - \frac{6}{\beta_n^2} + \frac{6}{\beta_n^4} - \frac{4716}{5\beta_n^6} + \frac{3902418}{70\beta_n^8}\right], \qquad n \geq 1 \tag{18}$$

where $\beta_n = \pi(4n+1)$.

The function φ_n accounts for the effects of the parameters: δ_n, τ, Bi. For limiting values of the parameter Bi it reduces to

$$\varphi_n = \tanh(\delta_n \tau), \qquad Bi \to \infty \tag{19}$$

and

$$\varphi_n = \coth(\delta_n \tau), \qquad Bi \to 0. \tag{20}$$

For all $0 < Bi < \infty$ and for all values $\tau > 0.72, \tanh(\delta_n \tau) = 1$ for all $n \geq 1$. Therefore $\phi_n = 1$ for $n \geq 1$.

When $\tau_1 > 0.72, \tanh(\delta_n \tau) = 1$, $\phi_n = 1$ for all $0 < Bi < \infty$, therefore $B_n = 1$ for $n \geq 1$. These characteristics lead to the following flux tube solutions.

IV. Spreading Resistance Solutions

A. Flux Tubes

The general compound disk solution reduces to the flux tube solutions, as shown in Fig. 4(c) and presented by [16]: for $\mu = -1/2$

$$\psi = \frac{8}{\pi\epsilon}\sum_{n=1}^{\infty}\frac{J_1(\delta_n \epsilon)\sin(\delta_n \epsilon)}{\delta_n^3 J_0^2(\delta_n)} \tag{21}$$

and for $\mu = 0$

$$\psi = \frac{16}{\pi\epsilon}\sum_{n=1}^{\infty}\frac{J_1^2(\delta_n \epsilon)}{\delta_n^3 J_0^2(\delta_n)}. \tag{22}$$

The two flux tube solutions have been correlated by Negus and Yovanovich [9] over a wide range of the parameter $\epsilon, (0 < \epsilon \leq 0.9)$. They reported for $\mu = -1/2$

$$\psi^T = 1 - 1.40978\epsilon + 0.34406\epsilon^3 + 0.04305\epsilon^5 + 0.02271\epsilon^7 \tag{23}$$

and for $\mu = 0$

$$\psi^q = 1.08076 - 1.41042\epsilon + 0.26604\epsilon^3 - 0.00016\epsilon^5 + 0.058\,266\epsilon^7 \tag{24}$$

where the superscripts T and q denote the equivalent isothermal and isoflux solutions respectively. For small values of ϵ the thermal spreading parameter for the isoflux boundary condition is approximately 8% greater than the spreading parameter for the isothermal boundary condition.

B. Isotropic Finite Disks

The dimensionless spreading resistance for isotropic $\kappa = 1$ finite disks $\tau_1 < 0.72$ with negligible thermal resistance at the heat sink interface $Bi = \infty$, as presented in Fig. 4(b), is given by the following solutions: for $\mu = -1/2$

$$\psi = \frac{8}{\pi\epsilon}\sum_{n=1}^{\infty}\frac{J_1(\delta_n \epsilon)\sin(\delta_n \epsilon)}{\delta_n^3 J_0^2(\delta_n)}\tanh(\delta_n \tau) \tag{25}$$

and for $\mu = 0$

$$\psi = \frac{16}{\pi\epsilon}\sum_{n=1}^{\infty}\frac{J_1^2(\delta_n \epsilon)}{\delta_n^3 J_0^2(\delta_n)}\tanh(\delta_n \tau). \tag{26}$$

If the external resistance is negligible $Bi \to \infty$, the temperature at the lower face of the disk is assumed to be isothermal. The solutions for isoflux, $\mu = 0$, heat source and isothermal base temperature were given by [5] for

1) the centroid temperature;
2) the area-average contact area temperature.

C. Correlation Equations for $\mu = 0$ and $0 < Bi < \infty$

The solution for the isoflux boundary condition with external thermal resistance was recently re-examined by [25], [26]. They nondimensionalized the constriction resistance based on the centroid and area-average temperatures using the square root of the contact area as recommended by [12], and compared the analytical results against the numerical results reported by [24] over the full range of the independent parameters: Bi, ϵ, τ. Nelson and Sayers [24] also chose the square root of the contact area to report their numerical results. The analytical and numerical results were reported to be in excellent agreement.

Song *et al.* [25] and Lee *et al.* [26] developed simple closed-form expressions for the dimensionless constriction resistance based on the area-average and centroid temperatures. They defined the dimensionless constriction parameter as $\psi = \sqrt{\pi} k a R_c$ and gave the following expressions for the area-average temperature:

$$\psi_{\text{ave}} = \tfrac{1}{2}(1-\epsilon)^{3/2}\varphi_c \tag{27}$$

and for the centroid temperature

$$\psi_{\max} = \frac{1}{\sqrt{\pi}}(1-\epsilon)\varphi_c \tag{28}$$

with

$$\varphi_c = \frac{Bi \tanh(\delta_c \tau) + \delta_c}{Bi + \delta_c \tanh(\delta_c \tau)} \tag{29}$$

and

$$\delta_c = \pi + \frac{1}{\sqrt{\pi \epsilon}}. \tag{30}$$

Song *et al.* [25] and Lee *et al.* [26] reported that the above approximations are within ±10% of the analytical results and the numerical results of [24]. They did not, however, indicate where the maximum errors occur.

D. Single Layer on Flux Tube

In a recent publication by [8] solutions were presented for the effect of multiple layers on the thermal constriction resistance of a circular heat source which is subject to either

1) uniform heat flux;
2) equivalent isothermal heat flux.

The solution for an isoflux circular heat source in perfect contact with a single layer of thickness t_1 and thermal conductivity k_1 which is placed in perfect thermal contact with an isotropic flux tube whose thermal conductivity is k_2 as shown in Fig. 3(c) is presented next. The dimensionless spreading resistance which is defined as $\psi = 4k_1 a R_c$ is given by

$$\psi = \frac{16}{\pi \epsilon} \sum_{n=1}^{\infty} \frac{J_1^2(\delta_n \epsilon)}{\delta_n^3 J_0^2(\delta_n)} \phi. \tag{31}$$

The effects of the layer and substrate thermal conductivities and the layer thickness are determined by the parameter

$$\phi = \frac{(1 + k_2/k_1) + (1 - k_2/k_1)\exp(-2\delta_n \epsilon \tau_1)}{(1 + k_2/k_1) - (1 - k_2/k_1)\exp(-2\delta_n \epsilon \tau_1)} \tag{32}$$

where $\epsilon = a/b$ is the relative contact radius, and $\tau_1 = t_1/a$ is the relative layer thickness. The parameter δ_n are the roots of $J_1(\delta_n) = 0$ and they are computed quickly and accurately by means of the modified Stokes approximation given above. The parameter ϕ is clearly equal to one when $k_2/k_1 = 1$ and when the product $\epsilon\tau_1 \geq 0.72$. This solution then approaches the flux tube solution developed for an isotropic flux tube whose thermal conductivity is k_1 (Fig. 4).

E. Single Layer on Half-Space

Dryden [19] obtained the solution for the equivalent isothermal heat flux distribution

$$q(r) = \frac{Q}{2\pi a (a^2 - r^2)^{1/2}}. \tag{33}$$

He used the Hankel transform to obtain the temperature distributions within the layer and the substrate. The area-average temperature $\overline{T}_c$ of the contact area was obtained and by means of the definition $R_c = \overline{T}_c / Q$ he obtained the expression for the constriction resistance which is reported below in a modified form

$$R_c = \frac{1}{\pi k_1 a} \int_0^{\infty} \left[\frac{1 + K\exp(-2\zeta t_1/a)}{1 - K\exp(-2\zeta t_1/a)} \right] J_1(\zeta)\sin(\zeta) \frac{d\zeta}{\zeta^2} \tag{34}$$

where the thermal conductivity parameter K is defined as

$$K = \frac{1 - \dfrac{k_2}{k_1}}{1 + \dfrac{k_2}{k_1}}. \tag{35}$$

The range of this parameter is [−1, 1]. It has the values −1, 0, 1 corresponding to the values $k_2/k_1 = \infty$, 1, 0, respectively. The function that appears within the square brackets accounts for the effects of the thermal conductivity ratio k_2/k_1 and the relative thickness of the layer t_1/a.

For $k_2 = k_1, K = 0$, the solution reduces to the well-known problem of an isothermal contact area situated on the surface of an isotropic half-space of thermal conductivity k_1 [21], [27] whose solution is

$$R_c = \frac{1}{\pi k_1 a} \int_0^{\infty} J_1(\zeta)\sin(\zeta)\frac{d\zeta}{\zeta^2} = \frac{1}{4k_1 a}. \tag{36}$$

If $k_1 \neq k_2$ and $t_1/a \to 0$, then the solution reduces to

$$R_c = \frac{1}{\pi k_2 a} \int_0^{\infty} J_1(\zeta)\sin(\zeta)\frac{d\zeta}{\zeta^2} = \frac{1}{4k_2 a}. \tag{37}$$

Dryden [19] proposed two simple expressions for thin and thick layers for the general case $k_1 \neq k_2$.

The spreading resistance for thin layers, $0 < t/a \leq 0.10$, is

$$R_c = \frac{1}{4k_2 a} + \frac{1}{\pi k_1 a}\left(\frac{t_1}{a}\right)\left[1 - \left(\frac{k_1}{k_2}\right)^2\right] \tag{38}$$

which consists of two terms. The first term is the spreading resistance within the substrate and the second term is a correction factor that accounts for the effects of the relative layer thickness and the thermal conductivity ratio.

The spreading resistance for thick layers, $2 < t_1/a < \infty$, is

$$R_c = \frac{1}{4k_1 a} - \frac{1}{2\pi k_1 a}\left(\frac{a}{t_1}\right)\ln\left(\frac{2}{1 + k_1/k_2}\right) \tag{39}$$

where the first term is the constriction resistance within the layer and the second term is the correction factor due to the relative layer thickness and the conductivity ratio.

In the intermediate range, $0.1 < t_1/a < 2$, the full integral solution must be used. It is relatively easy to obtain numerical values for all values of k_1/k_2 in this range by the use of Computer Algebra Systems such as Maple [28], Mathematica [29]), and MATLAB [30].

F. Isoflux Contact on Layer on Half-Space

Hui and Tan [1] used the separation of variables method combined with the Hankel transform to obtain expressions for the temperature distributions within a finite circular cylinder of radius b and thickness t_1 and thermal conductivity k_1 which is in perfect contact with an isotropic half-space of thermal conductivity k_2 as shown in Fig. 3(c). They considered the isoflux boundary condition $q = Q/\pi a^2$ over the circular source area of radius a which is located at the free end of the cylinder. The boundary condition outside the contact area is adiabatic and so is the lateral boundary of the cylinder. The free surface of the half-space is assumed to be adiabatic. They also report the special case where the radius of the finite thickness cylinder becomes infinitely large relative to the contact radius. This corresponds to an isoflux circular contact situated on an infinite layer which is in perfect contact with a half-space as shown in Fig. 3(d). They presented expressions for the heat source temperature rise and the area-average heat source temperature rise.

The temperature rise distribution within the contact area is

$$T(r)=\frac{qa}{k_1}\int_0^\infty\left[\frac{k_1+k_2\tanh(\zeta t_1/a)}{k_2+k_1\tanh(\zeta t_1/a)}\right]\frac{J_1(\zeta)}{\zeta}J_0\left(\zeta\frac{r}{a}\right)\,d\zeta. \tag{40}$$

The area-average temperature rise of the heat source area is

$$\overline{T}=\frac{qa}{k_2}\left\{\frac{8}{3\pi}\left(\frac{k_2}{k_1}\right)^2+2\left[1-\left(\frac{k_2}{k_1}\right)^2\right]\cdot\int_0^\infty\frac{J_1^2(\zeta)d\zeta}{[1+k_1/k_2\tanh(\zeta t_1/a)]\zeta^2}\right\}. \tag{41}$$

The spreading resistance can be obtained from the area-average temperature expression through $R_c=\overline{T}/q\pi a^2$. Since the dimensionless spreading resistance parameter is defined as $\psi=4k_2aR_c$, it takes the form

$$\psi^q=\frac{32}{3\pi^2}\left(\frac{k_2}{k_1}\right)^2+\frac{8}{\pi}\left[1-\left(\frac{k_2}{k_1}\right)^2\right]\cdot\int_0^\infty\frac{J_1^2(\zeta)\,d\zeta}{[1+k_1/k_2\tanh(\zeta t_1/a)]\zeta^2}. \tag{42}$$

If $k_2=k_1$, the above expression reduces to the well-known value [21], [27]

$$\psi^q=\frac{32}{3\pi^2}=1.08076. \tag{43}$$

Hui and Tan [1] did not provide simple algebraic expressions for thin and thick layers. It is therefore necessary to evaluate the above infinite integral numerically. Computer Algebra Systems provide convenient means for obtaining accurate values of ψ^q.

G. Isoflux, Equivalent Isothermal and Isothermal Solutions

The problem of finding the thermal constriction resistance for a circular contact area on an infinite isotropic layer of thickness t_1 and thermal conductivity k_1 placed in perfect contact with an isotropic half-space of thermal conductivity k_2 was undertaken by [11]. The solutions were obtained with the application of the Hankel transform method for flux specified boundary conditions and with a novel technique of linear superposition for the mixed boundary condition (isothermal contact area and zero flux outside the source area). Their results are presented below.

For the isoflux boundary condition they reported the result for $\psi^q=4k_1aR_c$

$$\psi^q=\frac{32}{3\pi^2}+\frac{8}{\pi^2}\sum_{n=1}^{\infty}(-1)^n\mathcal{K}^nI_q. \tag{44}$$

The first term is the dimensionless isoflux constriction resistance of an isotropic half-space of thermal conductivity k_1 and the second term accounts for the effect of the relative layer thickness and the relative thermal conductivity. The thermal conductivity parameter K is defined as

$$\mathcal{K}=\frac{1-\kappa}{1+\kappa} \tag{45}$$

with $\kappa=k_1/k_2$. The layer thickness-conductivity parameter I_q is defined as

$$I_q=\frac{1}{2\pi}\left\{2\sqrt{2(\gamma+1)}\,E(\sqrt{2/(\gamma+1)})-\frac{\pi}{2\sqrt{2\gamma}}I_\gamma-2\pi n\tau_1\right\} \tag{46}$$

with

$$I_\gamma=\left(1+\frac{0.09375}{\gamma^2}+\frac{0.0341797}{\gamma^4}+\frac{0.00320435}{\gamma^6}\right). \tag{47}$$

The relative layer thickness is $\tau_1=t_1/a$ and the relative thickness parameter is

$$\gamma=2n^2\tau_1^2+1. \tag{48}$$

The special function $E(\cdot)$ is the complete elliptic integral of the second kind [31]. The following approximations of the complete and complementary elliptic integrals of the second kind are provided to simplify the computational effort.

The complete elliptic integral is

$$E(k)=\frac{\pi/2}{1+k_1}\left[1+\frac{k_1^2}{4}+\frac{k_1^4}{64}+\frac{k_1^6}{256}\right] \tag{49}$$

where the parameter k_1 is defined as

$$k_1=\frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}. \tag{50}$$

This approximation provides 6 digit accuracy everywhere except at $k=1$ where the error is approximately 0.3%.

The complementary elliptic integral is

$$E(\sqrt{1-k^2})=\frac{\pi}{4}(1+k)\left[1+\frac{p^2}{4}+\frac{p^4}{64}+\frac{p^6}{256}\right] \tag{51}$$

where $p=(1-k)/(1+k)$. This approximation provides six-digit accuracy everywhere except at $k=0$ where the error is approximately 0.3%.

For the equivalent isothermal flux boundary condition they reported the result for $\psi_{ei}=4k_1aR_c$

$$\psi_{ei}=1+\frac{8}{\pi}\sum_{n=1}^{\infty}(-1)^n\mathcal{K}^nI_{ei} \tag{52}$$

where as discussed above the first term represents the dimensionless constriction resistance of an isothermal source area on an isotropic half-space of thermal conductivity k_1 and the second term accounts for the effect of the relative layer thickness and the relative thermal conductivity. The thermal conductivity parameter $\mathcal{K}$ is defined above. The relative layer thickness parameter I_{ei} is defined as

$$I_{ei}=\left[\sqrt{1-\beta^{-2}}(\beta-\beta^{-1})+\tfrac{1}{2}\sin^{-1}(\beta^{-1})-2n\tau_1\right] \tag{53}$$

with $\tau_1 = t/a$ and

$$\beta = n\tau_1 + \sqrt{n^2\tau_1^2 + 1}. \tag{54}$$

For the isothermal source area [11] developed a correlation equation for their numerical results. They reported $\psi^T = 4k_1 a R_c$ in the form

$$\psi^T = F_1 \tanh F_2 + F_3 \tag{55}$$

where

$$F_1 = 0.49472 - 0.49236\kappa - 0.00340\kappa^2 \tag{56}$$

and

$$F_2 = 0.28479 + 1.3337\tau + 0.06864\tau^2 \quad \text{with} \quad \tau = \log_{10}\tau_1 \tag{57}$$

and

$$F_3 = 0.49300 + 0.57312\kappa - 0.06628\kappa^2 \tag{58}$$

where $\kappa = k_1/k_2$. The correlation equation was developed for resistive layers: $0.01 \le \kappa \le 1$ over a wide range of the relative thickness: $0.01 \le \tau_1 \le 100$. The maximum relative error associated with the correlation equation is approximately 2.6% at $\tau_1 = 0.01$ and $\kappa = 0.2$. Numerical results for $\psi^q, \psi_{ei}, \psi^T$ for a range of values of τ_1 and κ were presented in tabular form for comparison. They found that the values for $\psi^q > \psi_{ei}$ and that $\psi_{ei} \le \psi^T$. The maximum difference between ψ^q and ψ^T was approximately 8%. They found that $\psi_T > \psi_{ei}$ for very thin layers: $\tau_1 \le 0.1$ and for $\kappa \le 0.1$; however, the differences were less than 8%. For most applications the equivalent isothermal flux results and the true isothermal results are similar.

V. Bounds on Total Thermal Resistance

Upper and lower bounds on the total resistance of the general case shown in Fig. 1(a) will be proposed based on the results presented above. The actual resistance will lie between the upper and lower bounds which will be close in most applications.

The upper bound can be determined from

$$R_{\text{upper bound}} = \frac{t}{k_1\pi b^2} + \frac{\psi(\mu = 0)}{4k_1 a} + \frac{1.0808}{4k_2 b} \tag{59}$$

and the lower bound by

$$R_{\text{lower bound}} = \frac{t}{k_1\pi b^2} + \frac{\psi(\mu = -1/2)}{4k_1 a} + \frac{1}{4k_2 b}. \tag{60}$$

In the above two expressions the spreading parameter $\psi(\mu)$ is determined by means of (12) with (13)–(16). For the problem shown in Fig. 1(a) $\kappa = 1$, therefore $\phi_n = 0$, and $B_n = -\varphi_n$. The relationship given by (17) is replaced by (20) for the upper bound, and by (19) for the lower bound. The largest uncertainty in the estimate of the spreading resistance will occur when $t/b \to 0$. In this limit, the second term in the above two relationships becomes negligible. When $t/b > 0.72$ as shown in Fig. 1(d), $\varphi_n = 1$ for all $n \ge 1$. The difference between the upper and lower bounds will be less than 8% which occurs when $t/b = 0$.

VI. Conclusion

A review of the papers that present solutions for the effect of single layers on the thermal spreading resistance of a circular heat source that is subjected to various heat flux distributions has been presented. The review covers solutions for compound disks, for heat flux tubes and for infinite layers in perfect thermal contact with a half-space. It is shown that the compound disk solution presented by [2] can be used to calculate the spreading resistance for all cases including the flux tube and half-space problems.

Approximations proposed by various researchers are presented for quick calculations of the spreading resistance. Upper and lower bounds on the total thermal resistance are proposed for the spreader-heat sink problem which can be applied to the diamond spreader-copper heat sink system. The maximum difference between the upper and lower bounds on the total resistance will be less than 8% for most applications.

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