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COMPREHENSIVE REVIEW OF NATURAL AND MIXED CONVECTION HEAT TRANSFER MODELS FOR CIRCUIT BOARD ARRAYS

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A review of the currently available models and correlations for natural and mixed convection heat transfer in vertical, uniformly heated parallel plate channels is presented. These results have been widely used for the thermal analysis of arrays of printed circuit boards contained in typical microelectronics and telecommunications equipment. This chronological presentation will begin with the well known work of Elenbaas[1] and describe the subsequent research for natural and mixed convection cooled isothermal and isoflux flat plate channels. All available heat transfer models will be presented and compared with numerical and empirical results from the current literature. In addition, the effects of many of the simplifying assumptions used in these models and correlations will be evaluated based on research presented by a number of authors.

1. Introduction

One of the most common vehicles for packaging multiple circuit cards in microelectronics and telecommunications equipment is the flowthrough module, a vertically oriented, parallel array of boards mounted within a vented enclosure, as shown in Fig. 1. Flowthrough modules are used for a wide range of applications, from simple systems containing two or three circuit boards, to more complex units, in which many of these modules can be combined as repeated units, making them the building blocks of many electronic equipment designs. Any thermal analysis of this system can therefore focus on these self-contained units rather than on larger cabinets containing multiple circuit board racks.

Circuit boards in typical flowthrough modules are cooled primarily by buoyancy-induced airflow that enters inlet vents at the bottom of the module and exits from openings at the top. This natural convection-cooled enclosure design can be used successfully in most applications; however, in cases where air velocities greater than the 0.3–0.5 m/s attainable with buoyancy driven flow are required, a small fan or blower can be positioned at either the intake or exhaust vents.

With the continuing trend of miniaturization of microelectronic circuitry and because of the limited cooling capacity associated with natural convection air cooling, optimal thermal design of these modules is critical to ensure adequate system performance and reliability. The most effective way to achieve an optimized design prior to prototyping or production is through the use of quick and accurate analytical heat transfer models during the design process. Using analytical models for trade-off and "what-if" studies can help determine optimal board spacing, power levels and component locations, and assess the need for fan-assisted flow, prior to prototyping or production.

In response to this need for analytical models, many researchers, starting with Elenbaas[1] in 1942 and continuing on through to the present day, have performed experimental and numerical simulations and developed analytical models and correlations for natural and mixed convection in vertical channels. However, because of the number of researchers involved in this work — over 50 publications — and

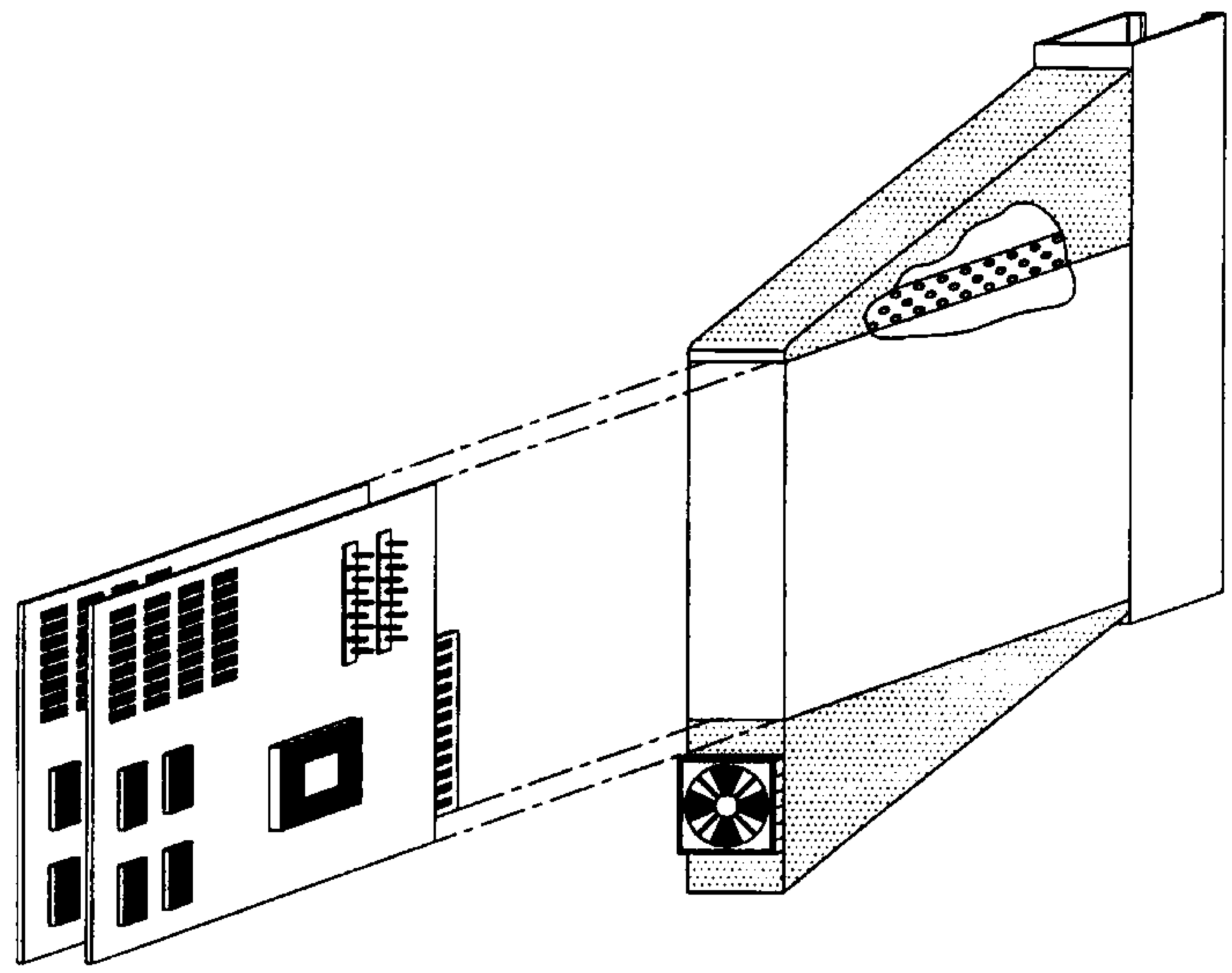

Fig. 1. Typical microelectronic flowthrough module enclosure.

the large time span over which the research has been done — over 50 years — as well as the widely-varying analysis techniques found in each work, it can be difficult for the engineer to select a model from the literature and be confident that the more appropriate one has been chosen.

A number of excellent review articles and chapters on air cooling for electronic applications that discuss the particular problem of the flowthrough module are currently available in the literature. Jaluria[2] presents a brief review of the major research efforts on this topic but concentrates most of his effort on numerical modeling of transport from single and multiple surface-mounted heat sources. Johnson[3] describes the major models available for uniform heat flux channels and compares them to empirical data from the literature. In their exhaustive treatment of direct air cooling of electronic components, Moffat and Ortega[4] present a good review of the major work for the natural convection channel, although a good portion of their review is devoted to the effects of surface irregularities and channel obstructions on board temperatures and heat transfer. Incropera[5] presents a general review of convection in electronic equipment cooling, of which the special case of the vertical, uniformly-heated channel is a subset. The review of natural convection air cooling of electronics by Aihara[6] also gives an overview of some of the major works on this topic. However, none of these reviews includes an exhaustive treatment of all the relevant publications, including a full comparison of the available models with empirical and numerical data, where possible, and an analysis of the effects of the various modeling assumptions and simplifications.

The objective of this study is therefore to present a comprehensive review of all the pertinent models and correlations available for this problem and compare them with each other and the available experimental and numerical data. Natural and mixed convection studies will be reviewed separately, with the following topics being examined for each case: background review of the relevant non-dimensional parameters, review of uniform temperature and uniform heat flux models, and evaluation of the effects of the simplifications and assumptions used in many of the models.

2. Natural Convection

2.1. *Background*

The heat transfer results for most of the models and results in the current literature are presented in terms of the classical natural convection non-dimensional values, the Nusselt and Rayleigh numbers. For the particular case of the vertical isothermal channel, the functional dependence of the Nusselt number on the Rayleigh number was first identified by Elenbaas[1]:

$$Nu_b = f\left(Gr_b \cdot Pr \cdot \frac{b}{L}\right) \tag{1}$$

where the Grashof number was defined as

$$Gr_b = \frac{g\beta(T_w - T_0)b^3}{\nu^2}\,. \tag{2}$$

This resulted in the definition of the so-called Elenbaas or channel Rayleigh number for the isothermal case:

$$Ra_b = \frac{g\beta(T_w - T_0)b^4}{\nu^2 L} \cdot Pr\,. \tag{3}$$

For the corresponding isoflux case, the classical modified Rayleigh number can be converted to its channel equivalent in the same manner:

$$Ra_b^\star = \frac{g\beta q b^5}{k\nu^2 L} \cdot Pr\,. \tag{4}$$

Although many different Nusselt number definitions are presented in the literature, this study will focus on the following forms for the isothermal and isoflux cases. The local heat transfer rate along the flow direction is characterized by the local Nusselt number, defined as

$$Nu_b(x) = \frac{q(x)\,b}{k(T_w(x) - T_0)}\,. \tag{5}$$

For the particular case of uniform wall temperature boundary conditions, $T_w(x) = T_w$, the average, or overall Nusselt number is defined as

$$\overline{Nu}_b = \frac{Q\,b}{kA(T_w - T_0)}\,, \tag{6}$$

where Q is the total heat flow rate from the channel wall with surface area A. For the isoflux boundary, the overall Nusselt number is generally based on the maximum wall temperature at the end of the channel

$$Nu_b(L) = \frac{q\,b}{k(T_w(L) - T_0)}\,, \tag{7}$$

although some researchers have presented their results in terms of the mid-channel temperature, $T_w(L/2)$.

A number of simplifying assumptions are common to all of the models reviewed in this section. First, it is assumed that the channel width is much smaller than its depth, $b \ll W$, such that the flow field can be treated as two-dimensional. Radiation from the circuit boards to the surrounding enclosure or environment is also neglected and the channel walls are assumed to be uniformly heated, with either isothermal or isoflux boundary conditions. Finally, flow restrictions induced by common design elements found in actual flowthrough modules, such as card guides, flow divertors and protruding packages, are not considered in these models. The resulting, simplified channel configuration is shown in Fig. 2, where the physical dimensions and boundary conditions are described.

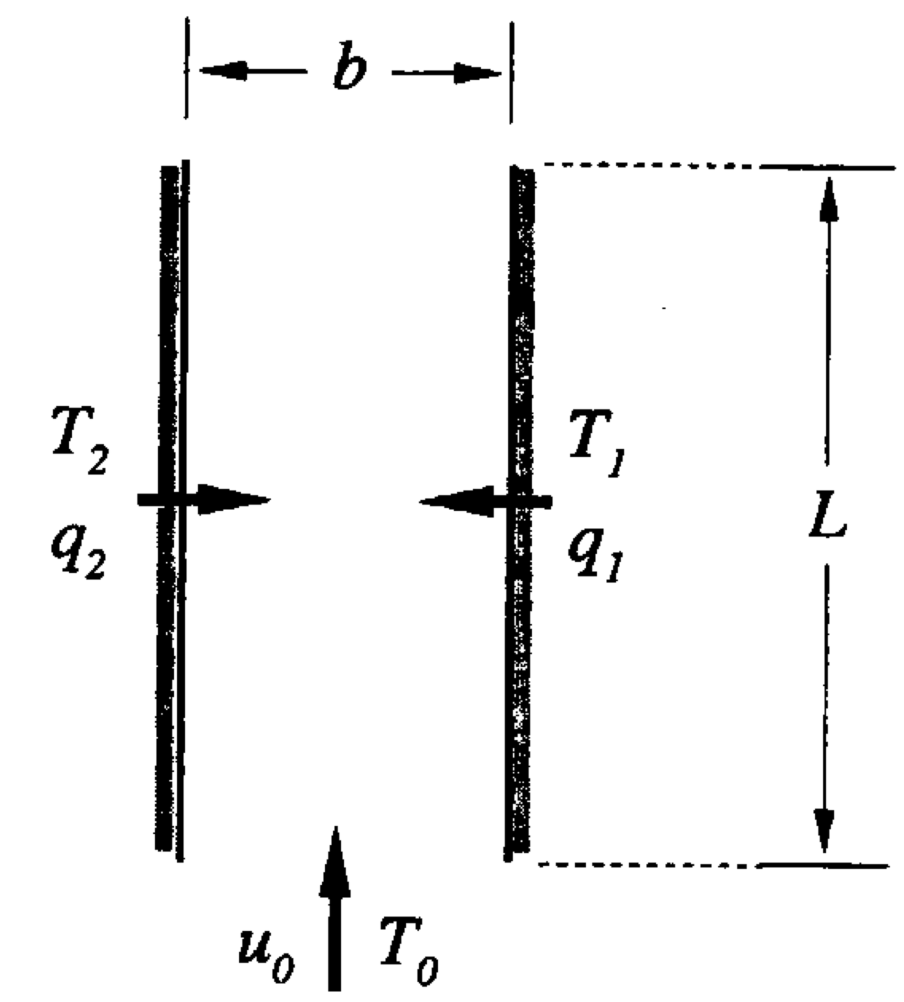

Fig. 2. Schematic of two-dimensional channel problem.

2.2. *Isothermal flat plate channel models*

Elenbaas[1] was the first researcher to present a detailed study of natural convection for the uniformly heated, vertical channel problem. Using two polished aluminum plates with embedded heater elements, he measured plate temperature using a centrally-located thermopile for this symmetrically-heated channel. Elenbaas also provided an analytical model for the average Nusselt number based on two proposed solutions to the simplified continuity, momentum and energy equations. These solutions were manipulated based on the available experimental results, as well as the behavior of the resulting expression for the limiting cases, $b \to 0$ and $b \to \infty$. The resulting expression, valid for the symmetrically-heated isothermal channel, is

$$\overline{Nu}_b = \frac{1}{24} Ra_b \left[1 - \exp\left(-\frac{35}{Ra_b}\right)\right]^{3/4}\,. \tag{8}$$

Bodoia and Osterle[7] followed Elenbaas[1] with a numerically-based study of developing flow, heat transfer between symmetrically-heated, isothermal plates in an effort to predict the channel length required to achieve fully-developed flow as a function of the channel width and wall temperature. These authors verify the limit of the Nusselt number for small Rayleigh number to be

$$\overline{Nu}_b = \frac{1}{24} Ra_b\,, \tag{9}$$

which corresponds to the limit of Elenbaas[1] for small Ra_b. They also identify the free convection result for a single flat plate as the limit for large Rayleigh numbers, proposing a relationship of the form:

$$\overline{Nu}_b = f(Pr) Ra_b^{1/4}, \tag{10}$$

where $f(Pr) = 0.68$ was recommended for $Pr = 0.7$.

In the first of a series of papers and reports concerning channel modeling, Aung[8] presents a closed-form solution for fully-developed flow in channels with unequal, uniform temperature boundary conditions. He introduces the temperature difference ratio, defined as follows for the isothermal case:

$$r_T = \frac{T_2 - T_0}{T_1 - T_0}, \tag{11}$$

where the limit $r_T = 1$ corresponds to symmetric heating and $r_T = 0$ describes a channel with one wall at the inlet temperature, $T_2 = T_0$. Using this ratio in the solution of the governing equation yields

$$\overline{Nu}_b = \left(\frac{4\, r_T^2 + 7\, r_T + 4}{90(1 + r_T)^2} \right) \overline{Ra}_b, \tag{12}$$

where the notation $\overline{Ra}_b$ refers to the Rayleigh number based on the average wall temperature, determined based on r_T and T_1 using

$$\overline{Ra}_b = \frac{g\beta(1 + r_T)(T_1 - T_0)b^4}{2\nu^2 L} \cdot Pr. \tag{13}$$

This expression can be shown to correspond to the Bodoia and Osterle[7] and Elenbaas[1] fully-developed expressions when $r_T = 1$.

In a companion paper, Aung *et al.*[9] present the results of a combined experimental and numerical study of heat transfer in developing flow channels with asymmetric heating. Their findings are plotted as temperature and velocity profiles and no attempts are made to provide models or correlations of their data.

Miyatake and Fujii[10] continued this research with their numerical analysis of channels with unequal, uniform temperature walls and they present an expression valid for $Pr = 0.7$ for the large Rayleigh number limit

$$\overline{Nu}_b = 0.58 \left(1 + 0.165\, r_T^{0.36}\right) Ra_b^{1/4}, \tag{14}$$

where Ra_b is calculated using T_1.

In the special case of a channel with one wall unheated and adiabatic, Miyatake and Fujii[11] present expressions for the limits of the average Nusselt number. For large Ra_b with a uniform velocity profile at the inlet and $Pr = 0.7$,

$$\overline{Nu}_b = 0.613 Ra_b^{0.25}, \tag{15}$$

while for the same limit with a parabolic inlet condition, the coefficient in the expression changes slightly:

$$\overline{Nu}_b = 0.627 Ra_b^{0.25}. \tag{16}$$

For small values of Ra_b, the authors predict that the solution will asymptotically approach the following relationship:

$$\overline{Nu}_b = \frac{1}{12} Ra_b. \tag{17}$$

For each of these expressions, Ra_b is calculated using the temperature of the heated wall.

Quintiere and Mueller[12] present an approximate analytical study of developing natural convection in parallel plate channels based on a linearization of the governing equations. The results of this work include models for fluid and wall temperature, velocity and pressure loss in series solution form. Although this work is one of the most complete theoretical analyses of the channel problem to date, the complexity of the resulting series expressions require the use of a computer program to generate solutions and may preclude its use as a rapid design tool.

In an effort to correlate the available air-cooled channel data for the full range of Ra_b, Raithby and Hollands[13] propose the following expression:

$$\overline{Nu}_b = 0.6 Ra_b^{1/4} \sum_{n=1}^{\infty} (-1)^{n+1} \times \frac{3}{(4n-1)(n-1)!} \left(\frac{C}{Ra_b} \right)^{n-1}, \tag{18}$$

where $C = 31$ for a symmetrically-heated channel and $C = 15.5$ for the case with one unheated, adiabatic wall.

In their finite element-based analysis, Ofi and Hetherington[14] present their results for the symmetrically-heated channel using temperature and velocity profiles. Based on their results, these authors also propose an empirically-based expression for the large Ra_b limit on the average Nusselt number:

$$\overline{Nu}_b = 0.699 Ra_b^{1/4}. \tag{19}$$

A simplified model for the average Nusselt number for the symmetrically-heated, isothermal channel is proposed by Churchill,[15] which uses his previously-published composite solution technique (Churchill and Usagi[16]). By combining the expressions for the limiting cases of small and large Ra_b and assuming a well-behaved transition, as demonstrated by previous experimental and numerical data, a model is developed which is valid for the full range of Ra_b:

$$\overline{Nu_b} = \left[\left(\frac{24}{Ra_b}\right)^{3/2} + \left(\frac{[1+(0.492/Pr)^{9/16}]^{4/9}}{0.75Ra_b^{1/4}}\right)^{3/2}\right]^{-2/3} . \tag{20}$$

Bar-Cohen and Rohsenow[17] modified and extended the Churchill composite solution based on existing data and proposed the following model for the isothermal case where $Pr = 0.7$:

$$\overline{Nu_b} = \left[\left(\frac{C}{Ra_b}\right)^2 + \left(\frac{1}{0.59Ra_b^{1/4}}\right)^2\right]^{-1/2} , \tag{21}$$

where $C = 24$ and 12 for the symmetric and one unheated, adiabatic wall cases, respectively.

For the channel with unequal, uniform temperature walls, Raithby and Hollands[18] recommend the following composite model:

$$\overline{Nu_b} = \left[\left(\frac{90(1+r_T)^2}{4r_T^2+7r_T+4}\cdot\frac{1}{\overline{Ra_b}}\right)^{1.9} + \left(\frac{1}{C\overline{Ra_b}^{1/4}}\right)^{1.9}\right]^{-1/1.9} , \tag{22}$$

where one of the recommended values for the coefficient, $C = 0.62$, is verified in a later publication by Martin *et al.*[19] based on previous analytical and experimental work. Both the Nusselt and Rayleigh numbers in this expression are based on the average wall temperature, as defined in Eq. (13) for $\overline{Ra_b}$. It is proposed that air properties be evaluated at the average film temperature, $(\overline{T_w} + T_0)/2$, except in the case of small Ra_b, where the average wall temperature, $\overline{T_w}$, should be used. In all cases, the thermal expansion coefficient, β, is determined using the absolute inlet temperature, $\beta = 1/T_0$.

In order to conclude this section, the models available for predicting the average Nusselt number for the full range of Rayleigh numbers, $1 < Ra_b < 10^5$, will be compared with a representative sample of the available experimental and numerical data. In Fig. 3 the equal, uniform wall temperature models of Elenbaas[1] and Churchill[15] are compared with the unequal wall temperature results of Raithby and Hollands,[18] where $r_T = 1$. The remaining models, Bar-Cohen

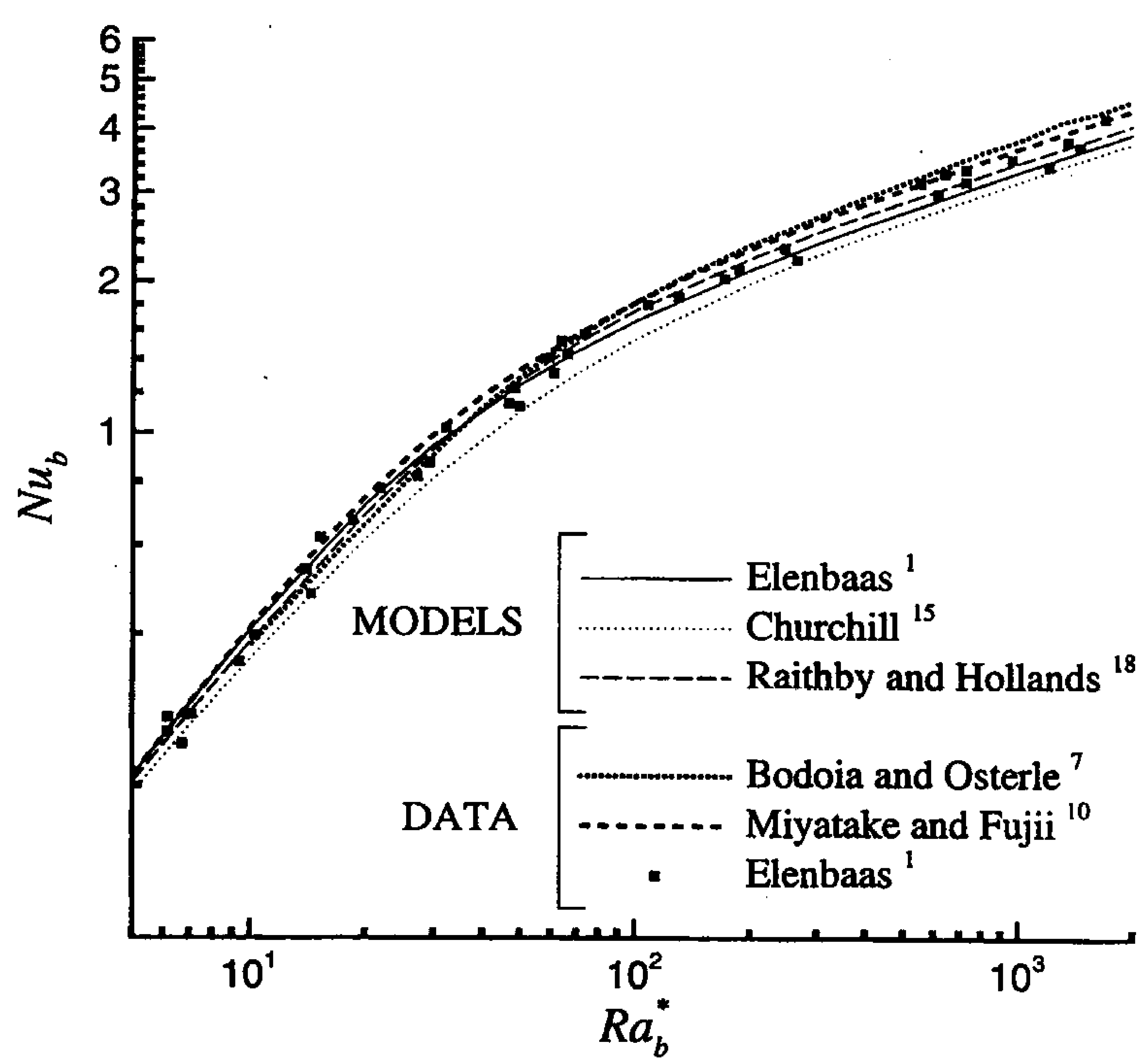

Fig. 3. Comparison of models for isothermal, symmetrically-heated flat plate channels with numerical and experimental data.

and Rohsenow[17] and Raithby and Hollands,[13] are not shown because they would obscure the results. However, each of these models fall between the lower bound of Churchill[15] and the upper bound of Elenbaas[1] and Raithby and Hollands,[18] for small and large Ra_b, respectively. Also plotted in Fig. 3 are the experimental data of Elenbaas,[1] the numerical results of Bodoia and Osterle[7] and the numerical data for $r_T = 1$ from Miyatake and Fujii.[10]

Figure 3 clearly demonstrates the close agreement between all of the models for the equal, uniform temperature boundary condition, with a maximum difference of less that 16% between the upper and lower bounds. These models also closely predict the available data, with an maximum difference of between 14% and 18% for $Ra_b > 10^3$. Although not examined explicitly, it can be inferred from this comparison that those models capable of incorporating unequal, uniform wall temperatures, Bar Cohen and Rohsenow[17] and Raithby and Hollands,[18] will remain accurate when $0 \leq r_T < 1$.

As a result, it can be concluded that any of these models are suitable for predicting heat transfer for isothermal, flat plate channels. In particular, the Churchill[15] model is recommended for problems involving fluids other than air where $Pr \neq 0.7$, the Bar-Cohen and Rohsenow[17] model for channels with one unheated, adiabatic wall, and the Raithby and Hollands[18] model for unequal, uniform wall temperature boundary conditions.

Other data sources not included in this comparison but available in the literature are the experimental results of Currie and Newman[20] and the the numerical data of Nakamura *et al.*[21] and Ofi and Hetherington.[14]

2.3. *Isoflux flat plate models*

The first well-documented research effort focused on the vertical channel problem with a uniform heat flux boundary condition was that of Sobel *et al.*[22] These authors correlated the results of their experimental measurements for air in a symmetrically-heated isoflux channel using a power law fit, where the Nusselt number is defined using the temperature at the midpoint of the plate:

$$Nu_{b(L/2)} = 0.666 {Ra_b^\star}^{1/5} . \qquad (23)$$

This expression is proposed to be valid for the range of modified Rayleigh numbers $5 \leq Ra_b^\star \leq 3500$.

Following their preliminary work on the isoflux channel with one unheated, adiabatic wall (Miyatake *et al.*[23]), Miyatake and Fujii[24] examined the vertical channel problem with unequal, uniform heat fluxes using both numerical and analytical techniques and developed a number of models and correlations for the local Nusselt number. Based on the use of a parabolic velocity profile for fully-developed flow, the local Nusselt number for small values of $Ra_b^\star$ is derived analytically as

$$Nu_{b(x)} = \left[\frac{1}{2} + (1+r_q)\left(\frac{x}{L}\left(\frac{24}{(1+r_q)Ra_b^\star}\right)^{1/2} - \frac{9}{70}\right)\right]^{-1} , \qquad (24)$$

where r_q is the heat flux distribution ratio, defined as

$$r_q = \frac{q_2}{q_1} . \qquad (25)$$

In the entrance region, where the boundary layers on each side of the channel develop independent of each other, two correlations of the numerical results for the local Nusselt number are presented based on the velocity profile at the inlet: for a uniform inlet velocity,

$$Nu_{b(x)} = 0.40 \left(\frac{1}{x/L}\right)^{1/2} ((1+r_q)Ra_b^\star)^{1/4} \qquad (26)$$

and for a parabolic inlet velocity profile,

$$Nu_{b(x)} = 0.697 \left(\frac{1}{x/L}\right)^{1/3} ((1+r_q)Ra_b^\star)^{1/6} . \qquad (27)$$

Miyatake and Fujii[24] also develop a correlation similar in form to that used by Elenbaas[1] for the full range of $Ra_b^\star$, for $Pr = 0.7$:

$$Nu_{b(x)} = \frac{1}{(x/L)}\left(\frac{Ra_b^\star}{24(1+r_q)}\right)^{1/2} \times \left[1 - \exp\left(-\frac{2.84(1+r_q)^{3/4}(x/L)^{0.6}}{{Ra_b^\star}^{0.3}}\right)\right] , \qquad (28)$$

which is valid in the range $0 \leq r_q \leq 2$.

The composite solution technique introduced by Churchill and Usagi[16] and used by Churchill[15] in his isothermal model, Eq. (20), is also applied by this author for the isoflux, symmetrically-heated channel

(Churchill[15]):

$$Nu_b(L/2) = \left[\left(\frac{12}{Ra_b^\star}\right)^{3/2} + \left(\frac{[1+(0.437/Pr)^{9/16}]^{4/9}}{0.75 Ra_b^{\star 1/4}}\right)^{3/2}\right]^{-2/3}, \quad (29)$$

where changes to the isothermal model are limited to changes in two of the coefficients and the use of the isoflux definitions of the Nusselt and Rayleigh numbers.

Through their experimental tests for the isoflux symmetrically-heated channel, Wirtz and Stutzman[25] produced empirical heat transfer data for a wide range of $Ra_b^\star$ numbers. A corresponding model that predicts the Nusselt number at the channel exit based on these measured values is developed using the Churchill and Usagi[16] composite solution technique

$$Nu_b(L) = \left[\left(\frac{1}{0.114 Ra_b^{\star 1/2}}\right)^3 + \left(\frac{1}{0.577 Ra_b^{\star 1/5}}\right)^3\right]^{-1/3}. \quad (30)$$

Bar-Cohen and Rohsenow[17] present a similar expression but chose to define their Nusselt number based on the midpoint temperature

$$Nu_b(L/2) = \left[\left(\frac{1}{C Ra_b^{\star 1/2}}\right)^2 + \left(\frac{1}{0.73 Ra_b^{\star 1/5}}\right)^2\right]^{-1/2}, \quad (31)$$

where $C = 0.289$ for the symmetrically-heated channel and $C = 0.408$ for the channel with one unheated, adiabatic wall.

Also defined in terms of midpoint temperature is the model of Raithby and Hollands[18] for channels with unequal, uniform heat fluxes

$$\overline{Nu}_b(L/2) = \left[\left(\frac{1}{0.29 \overline{Ra}_b^{\star 1/2}}\right)^{3.5} + \left(\frac{1}{0.67 \overline{Ra}_b^{\star 1/5}}\right)^{3.5}\right]^{-1/3.5}, \quad (32)$$

where $\overline{Nu}_b(L/2)$ and $\overline{Ra}_b^\star$ are defined using the average heat flux, $\overline{q} = (q_1 + q_2)/2$.

In an extensive numerical analysis of the isoflux channel problem, Aihara and Maruyama[26] examine the effects of the temperature dependence of the fluid properties on the heat transfer. In the first stage of the analysis, they perform numerical simulations based on constant property assumptions and correlate their results in terms of a local Nusselt number:

$$\frac{1}{Nu_b(x)} = \frac{1}{2}\left(\frac{\sqrt{6}}{\phi} + 0.48\right) \times \left[1 - \exp\left(-\frac{124.7}{\phi(2.09 + Pr^{-1/2})Pr^{0.046}}\right)\right], \quad (33)$$

where

$$\phi = \frac{L}{x}\left(\frac{Ra_b^\star}{32}\right)^{1/2}\left[1 - \frac{0.035 Ra_b^{\star 1/4}}{Pr^{1/3}}\left(1 - \frac{x}{L}\right)\right]. \quad (34)$$

In the second stage of their research, Aihara and Maruyama[26] conclude that for air-cooled channel configurations, the constant and variable properties solutions differ by less than 25% and can be neglected in certain cases. A plot of the effects of the temperature dependence of properties on the local Nusselt number has also been provided by these authors for both air and oil.

Finally, for equal, uniform heat flux boundary conditions, Fujii *et al.*[27] propose the following modification to the correlation equation presented by Miyatake and Fujii[24] based on deviations noted between their numerical and experimental results and the model:

$$Nu_b(x) = \frac{1}{(x/L)}\left(\frac{Ra_b^\star}{48}\right)^{1/2} \times \left[1 - \exp\left(-\frac{5.72(x/L)}{Ra_b^{\star 0.33}}\right)\right]. \quad (35)$$

A number of conclusions concerning these models can be reached through their comparison with each other and with available experimental and numerical data for the full range $1 < Ra_b^\star < 10^5$, as presented in Fig. 4 for Nusselt number defined at the end of the channel, $x = L$, and in Fig. 5 for Nusselt number based on $L/2$. In Fig. 4, the equal, uniform heat flux boundary condition models of Wirtz and Stutzman,[25] Aihara and Maruyama[26] and Fujii *et al.*[27] and the unequal wall flux model of Miyatake

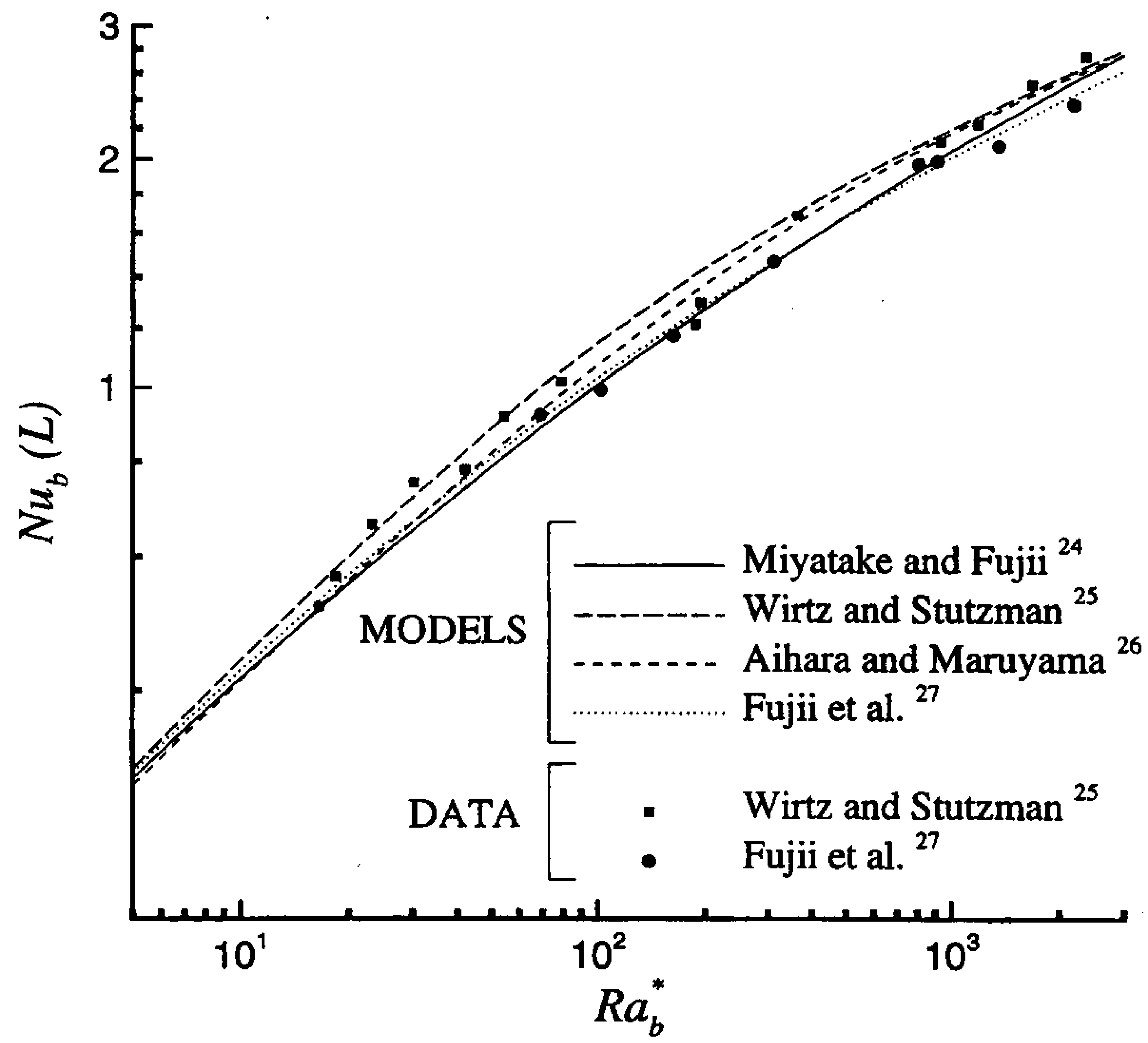

Fig. 4. Comparison of models for isoflux, symmetrically-heated flat plate channels with numerical and experimental data for Nusselt number evaluated at $x = L$.

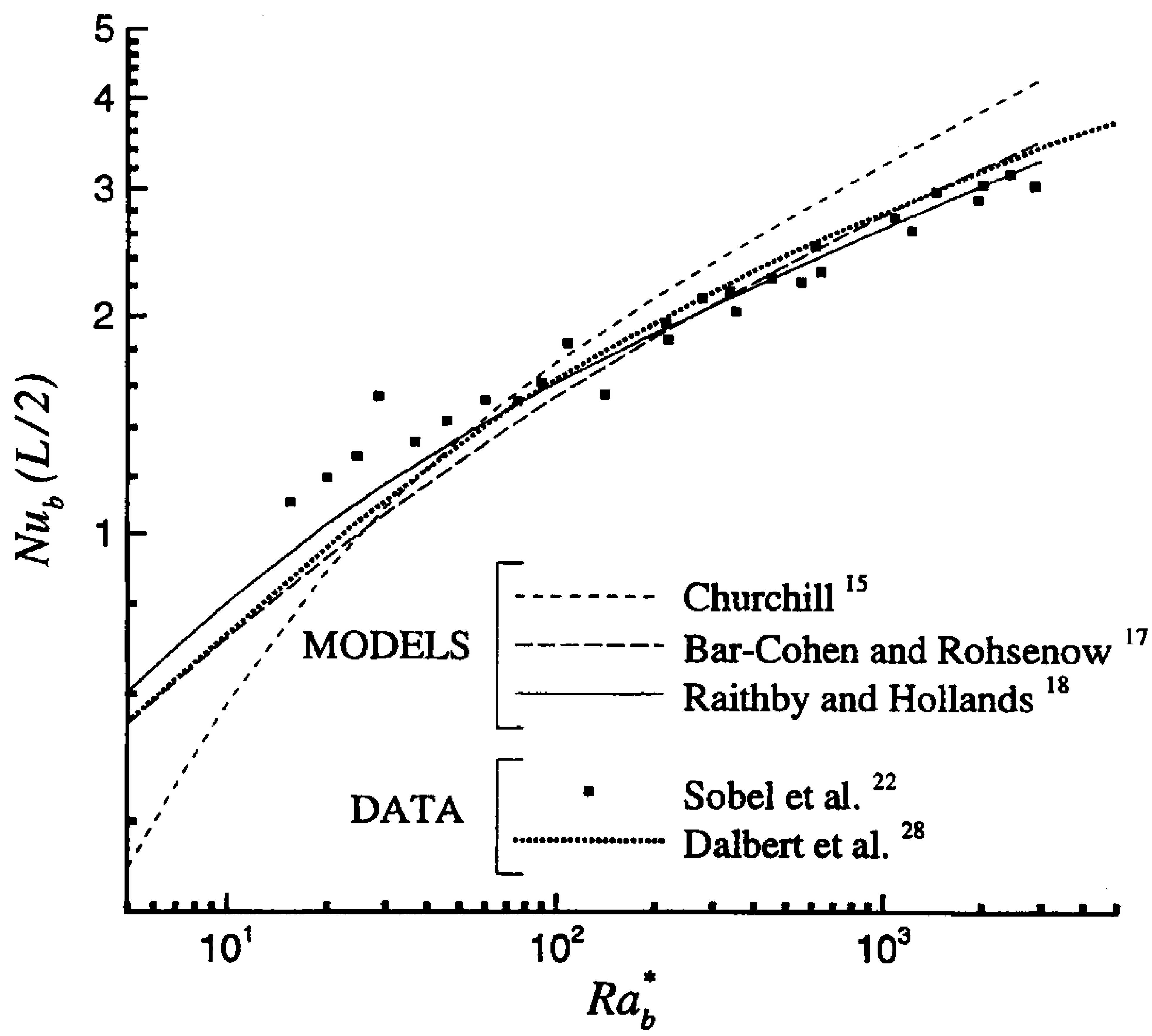

Fig. 5. Comparison of models for isoflux, symmetrically-heated flat plate channels with numerical and experimental data for Nusselt number evaluated at $x = L/2$.

and Fujii[24] with $r_q = 1$ are evaluated at $x = L$ and plotted with the experimental data of Wirtz and Stutzman[25] and Fujii *et al.*[27] for the isoflux, symmetrically-heated channel. Figure 5 presents the symmetric, isoflux models of Churchill[15] and Bar-Cohen and Rohsenow[17] and the unequal, uniform heat flux model of Raithby and Hollands[18] with $r_q = 1$ for midpoint plate temperatures compared with the experimental data of Sobel *et al.*[22] and the numerical results of Dalbert *et al.*[28]

In Fig. 4 the close agreement between all of the models and the available data for $Nu_b(L)$ is clearly demonstrated. The maximum difference between the models occurs at $Ra_b^\star \approx 100$, where Miyatake and Fujii,[24] the lower bound on the models, differs from the upper bound of Wirtz and Stutzman[25] by approximately 13%. Figure 5 shows the good agreement between the composite solution-based models of Bar-Cohen and Rohsenow[17] and Raithby and Hollands[18] and the numerical results of Dalbert *et al.*[28] The composite model of Churchill[15] varies significantly from the other models for both small and large $Ra_b^\star$, an indication that incorrect asymptote formulations were used in the model. As well, the empirical data of Sobel *et al.*[22] agrees closely with the models for large Rayleigh number, while for $Ra_b^\star < 100$ the data begins to deviate from the numerical and analytic results.

Ultimately, the choice of one of these models by the engineer will be based on his requirements for the solution. Models based on midpoint temperature, $x = L/2$, will predict heat transfer that is representative of the average value, while models using $x = L$ will give maximum, or worst case values. In particular, Miyatake and Fujii,[24] Fujii *et al.*[27] or Aihara and Maruyama[26] should be used to determine local values of $Nu_b(x)$, Aihara and Maruyama[26] is recommended for problems involving fluids other than air. The only model that is not recommended as a result of this study is Churchill,[15] due to the isoflux model's large inaccuracies for small and large $Ra_b^\star$.

2.4. *Evaluation of modeling assumptions*

Common to all of the models and correlations presented in the preceding sections are the following simplifying assumptions: the flow in the channel is two-dimensional and laminar; all radiation effects are insignificant and can be neglected; the channel walls are smooth and uniformly heated, with either an isothermal or isoflux boundary condition; no flow restrictions are present in the system; conduction in the board and into the upstream fluid does not occur; and no complex phenomenon, such as flow reversals, are present. Although the heat transfer solutions in the current literature accurately predict the empirical and numerical data gathered for these simplified geometries, the ability of these models to simulate real arrays of circuit boards has not been clearly established. The following section will examine each of these simplifying assumptions and briefly review the relevant publications that examine their effects on the solution of actual problems.

In a series of three papers, Ortega and Moffat[29,30] and Moffat and Ortega[31] experimentally investigate natural convection from an array of cubical heat sources mounted on the wall of a vertical channel, and model the heat transfer characteristics of this system using an adiabatic heat transfer coefficient. In this way, these authors propose an empirically-based correlation method that accounts for the hydrodynamic disturbances caused by the protruding elements, heat conduction through the board between adjacent packages and any thermal wake effects. A similar topic is addressed by Jaluria[32] in his numerical study of thermal wake effects for discrete heat sources mounted on a vertical, adiabatic board. Both of these authors comment on the inability of the current smooth-walled channel or plate models to accurately simulate circuit boards with surface mounted components.

Typical vented enclosures for electronic equipment often contain devices, such as screens, baffles or grilles, that can adversely affect system cooling by acting as a restriction to the free flow of air. Aung *et al.*[33] examined the effects of large protrusions in the channel, card guides and baffles on the heat transfer and they provide design charts and nomograms for each of these configurations. Birnbrier[34] presents an experimental study for uniformly-heated vertical arrays of circuit boards with large, surface mounted protrusions, where he notes a significant difference between his measured values of $Nu_b(L)$ and the smooth channel model of Aung *et al.*[9] The effects of perforated metal electro-magnetic containment (EMC) screens placed at the inlet and exit of the channel are studied experimentally by Kato *et al.*,[35] where they develop an empirical correlation for $\overline{Nu_b}$ as a function of Ra_b and the screen porosity. Finally, the effects of placing a horizontal baffle near the inlet or outlet of a vertically-oriented array of circuit boards is experimentally studied by O'Meara and Poulikakos.[36] Each of these studies suggest that the flow restrictions caused by the various elements in a typical vented enclosure design can be significant and should not be arbitrarily neglected.

Radiation effects for the vertical channel problem are examined numerically by Carpenter *et al.*,[37] who report that radiation can significantly contribute to the overall heat transfer for certain configurations. For symmetrically-heated channels, radiation through the channel openings to the surroundings can result in a 50% decrease in maximum wall temperature from that of convection alone for an aspect ratio $L/b \geq 5$. Even more significant reductions in wall temperature were noted for channels with one unheated wall. Sparrow *et al.*[38] report similar results from their numerical study, and conclude that for intermediate and large Rayleigh numbers, heat transfer for a channel with one unheated wall can be enhanced by 50%–75% by including radiation effects. In addition they found that for aspect ratios $10 < L/b < 100$, the radiation to the surroundings is about 10% of the total heat transferred by the system.

The assumption of uniform channel wall heating through the use of an isothermal boundary condition is examined numerically by Burch *et al.*[39] using a conjugate analysis of a channel with conductive walls. This work concludes that the influence of wall conduction may be significant for isothermal channel configurations with large Ra_b where the conductivity of the solid approaches that of the fluid.

Martin *et al.*[19] numerically studied the effects of upstream conduction from the channel inlet, as well as other inlet and outlet assumption effects. They concluded that the heat transfer at low Ra_b will depend on the shape of the intake and exhaust plenums and boundary conditions, as well as the aspect ratio of the channel, L/b. This effect is also noted in the numerical work of Ramanathan and Kumar[40] for isoflux channels, and both Martin *et al.*[19] and these authors suggest that the small Ra_b limit used by the composite solutions be modified for small aspect ratios, $L/b < 10$ for low Rayleigh number, i.e. $Ra < 1$.

In a combined experimental and numerical study of the channel problem where one wall is unheated, Sparrow *et al.*[41] described a flow reversal phenomenon that occurs at the trailing edge of the unheated plate for $Ra_b > 35000$. Their correlation of the average Nusselt number at this large Ra_b limit revealed the dependence of both the coefficient and the exponent on Pr. For the particular case of air, $Pr = 0.7$, the authors propose the following correlation:

$$\overline{Nu_b} = 0.667 Ra_b^{0.229} \tag{36}$$

where the power has changed slightly from the 0.25 proposed in the composite solutions.

3. Mixed Convection

3.1. *Background*

The previous definitions for both the Nusselt and Rayleigh numbers, Eqs. (3–7), continue to be valid for most of the studies in the literature to describe the natural convection component of the mixed convection, vertical channel problem. The remaining forced convection component is characterized by the Reynolds number, defined using the average entrance velocity:

$$Re_b = \frac{u_0 b}{\nu}. \tag{37}$$

Classical combined convection analysis recommends that when the Richardson number, Gr_b/Re_b^2, is less than 1, forced convection is the dominant heat transfer mode, and natural convection controls when $Gr_b/Re_b^2 \gg 1$. However, Yao[42] suggests that the following ratios of Grashof and Reynolds numbers be used to measure the relative importance of the natural convection component to the overall heat transfer: for uniform wall temperature channels, he recommends that buoyancy effects can be neglected when $Gr_b < Re_b$ and for the isoflux case when $Gr_b^2 < Re_b$. The discrepancy between these two methods may be due in part to the difference between the channel problem and problems involving a flat plate or an isolated body, for which the classical relationships were originally developed.

3.2. *Flat plate models*

Much of the preliminary research for assisted, mixed convection between vertical, parallel plates involved analytical modeling of the fully-developed flow problem. Tao[43] presents an analytic solution for the Nusselt number for fully-developed flow between symmetrically-heated parallel plates with uniform heat flux and extends his analysis by examining the full 3-D problem of flow in a rectangular duct. Agrawal[44] repeats this analysis of the mixed convection, rectangular duct problem but reduces the series solution of Tao[43] to a more simplified polynomial expression. A combined experimental and analytical study by Lauber and Welch[45] examined mixed convection cooling of an asymmetrically heated, annular region which, for small spacing-to-diameter ratios, can be modeled as parallel flat plates. These authors provide expressions for the Nusselt number at the channel exit for both the heated and unheated walls and report good agreement when they compare their model to their empirical data.

Developing flow and heat transfer in the entrance region of a parallel plate channel were modeled by Yao[42] for both the isothermal and isoflux symmetrically-heated problems. His solutions involve the combination of viscous boundary layer regions with an inviscid core solution, where heat conduction is restricted to the boundary layer region only. The resulting expressions for Nusselt number are as follows: for the isothermal case,

$$Nu_a = \left(\frac{Re_a}{2(x/L)}\right)^{1/2} \cdot \left(C_1 + 2C_2(x/L)\frac{Gr_a}{Re_a^2}\right) ; \qquad (38)$$

and for the isoflux channel,

$$Nu_a = \left(\frac{Re_a}{2(x/L)}\right)^{1/2} \cdot \left(\frac{1}{D_1 + D_2(2(x/L))^{3/2}(Gr_a/Re_a^2)}\right) , \qquad (39)$$

where the dimensionless values Nu_a, Re_a and Gr_a are defined using the half-channel width, $a = b/2$. Tabulated numerical values for the constants C_1, C_2, D_1 and D_2 are provided for $Pr = 0.01, 0.1, 1.0$ and 10.

In a series of three papers, Aung and Worku[46–48] present numerical mixed convection results for various channel configurations. These authors begin their research with an analytical study of fully developed, mixed convection for channels with unequal, uniform wall temperatures (Aung and Worku[46]) and present their results using temperature and velocity profiles plotted as functions of the temperature distribution ration r_T and Gr_b/Re_b. For the developing flow case, these authors perform numerical studies for the asymmetric isothermal boundary condition (Aung and Worku[47]) and the asymmetric isoflux case (Aung and Worku[48]). Once again, results are presented in the form of wall temperature, fluid temperature and velocity distributions as functions of the heat distribution ratios, r_T and r_q, and Gr_b/Re_b.

Habchi and Acharya[49] present a similar numerical analysis of this problem for both the symmetric isothermal channel and the channel with one unheated, adiabatic wall. Along with temperature and velocity profiles, their results include a plot of Nu_b vs. a Peclet number function, $(x/L)/Pe_b$. Numerical results for the symmetrically-heated isothermal channel are also presented using temperature and velocity profiles by Ingham *et al.*[50]

Jeng *et al.*[51] present their numerical results for the isothermal channel with unequal wall temperatures in terms of separate Nusselt numbers for each channel surface. These solutions are plotted as $(Nu_b - Nu_{b(Gr/Re \to \infty)})$ vs. $(x/L)/(D_1\,Re_b)$ and $(x/L)/(D_2\,Re_b)$ for the hot and cold walls, respectively. The coefficients D_1 and D_2 in these expressions are correlated in terms of Gr_b/Re_b and the fully-developed flow solution, $Nu_{b(Gr/Re \to \infty)}$ originally developed by Aung and Worku[46] is used. Therefore, given values of Gr_b/Re_b and r_T and using the plots presented in their paper, Nu_b for each channel surface can be predicted.

3.3. *Evaluation of modeling assumptions*

As in the natural convection modeling, many approximations have been made in the development of numerical and analytical heat transfer models for the fan-cooled, vented enclosure. However, since the effects of these simplifying assumptions on the solution have been clearly documented in a previous section for natural convection and it is assumed that the mixed convection solution behaves in a similar manner, these issues will not be discussed. Instead, two topics unique to the mixed convection cooling of vertical channels will be examined — flow reversal phenomenon and opposing convection.

In channels with strongly axisymmetric heating, i.e. $r_T < 0.5$, where the natural convection component is significant, i.e. $Gr_b/Re_b > 250$, many researchers, such as Aung and Worku[46–48], have noted a tendency for the fluid at the trailing edge of the cold wall to be drawn into the channel. This downward flow passes along the cool wall, reverses direction at some point in the channel, and exits via the buoyant plume from the hotter wall. Although the difficulties in modeling local Nusselt number on the cold wall due to the flow reversal phenomenon are obvious, it remains to be proven if this highly-localized effect will have an appreciable effect on the calculation of average values.

Opposed mixed convection, where the forced convection flow acts in the opposite direction of the buoyancy force, has been examined by a number of researchers as a heat transfer enhancement technique for channel cooling. Hallinan and Quereshi[52] demonstrate through their experimental results that through an earlier transition to turbulent behavior, opposed convection can be more effective than aided convection for high Reynolds number flows. Hamadah and Wirtz[53] present analytical solutions for fully developed, opposed convection for asymmetric isothermal, isoflux and combined boundary conditions in terms of correlation equations for Nusselt number at each wall.

4. Summary and Conclusions

A comprehensive review of heat transfer models for natural and mixed convection cooled flowthrough modules has been presented. For both the isothermal and isoflux, flat plate channel cooled by natural convection, many models and correlations are available to predict both the local and average Nusselt number as a function of the system geometry and fluid properties. Through a comparison of a number of these models with numerical and empirical data from the literature, it was demonstrated that for both the isothermal and isoflux channel problem, all of the models reviewed are capable of accurate temperature and heat flux predictions. With the exception of the Churchill[15] isoflux model, the maximum difference noted between the models was less than 11% and the average deviation between them and the data was approximately 6%. Therefore, the choice of model by the engineer can be based solely on his requirements of the solution, i.e. local versus average values, boundary conditions, etc., without concern for the accuracy of the result.

For the assisted, mixed convection channel, it was found that no models or empirically-based correlations are currently available that do not require the use of plots or tabulated data in order to predict Nu_b. Further research is required in this area to develop and validate mixed convection models for the vertical channel that are suitable for use by design engineers.

Many of the approximations and simplifications common to most of these models have been examined, such as the use of uniformly-heated flat plates and the effects of neglecting radiation and flow restrictions. The subsequent limitations of these models have also been discussed. Based on this review, it can be concluded that in order to meet the demands of the microelectronics and telecommunications industries for quick and accurate design tools, research into extending the currently available solutions to model more complex systems will certainly continue.

Nomenclature

a	channel half-spacing, $\equiv b/2$ (m)
b	channel spacing (pitch) (m)
C, D	miscellaneous constants
g	gravitational constant (m/s^2)
Gr_b	Grashof number, Eq. (2)
Gr_b^*	modified Grashof number, $\equiv (g\beta q b^4)/(k\nu^2)$
k	thermal conductivity (W/mK)
L	channel length (m)
Nu_b	Nusselt number, Eqs. (5–7)
Pr	Prandtl number
Q	total heat flow rate (W)
q	heat flux (W/m^2)
Ra_b	channel Rayleigh number, Eq. (3)
Ra_b^*	modified channel Rayleigh number, Eq. (4)
Re_b	Reynolds number, Eq. (37)
r_T, r_q	heat distribution ratios, Eqs. (11), (25)
T	temperature (°C)
u	axial velocity component (m/s)
W	channel depth (m)

Subscripts

0	inlet condition
w	at the wall
1, 2	at left and right wall, respectively

Greek Symbols

β	thermal expansion coefficient (1/K)
ν	kinematic viscosity (m^2/s)

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