FORMULATION OF ONE-DIMENSIONAL CONDUCTION WITH CONVECTION IN ORTHOGONAL CURVILINEAR COORDINATES: APPLICATION TO EXTENDED SURFACES

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ABSTRACT

A general equation in orthogonal curvilinear coordinates for heat transfer through extended surfaces(fins) is developed. The general solution for perfect contact at the fin base and end cooling is presented. The heat transfer rate through the fin is obtained and the fin resistance is defined. Several examples of the application of the general results are given. The temperature distribution, heat transfer rate and corresponding fin resistance are given for three coordinate systems: Cartesian, circular cylinder and spherical. Solutions for (i) insulated fin end and (ii) perfect contact at the fin end are also presented. It is shown that the well-known solutions (longitudinal fin of rectangular profile, circular annular fin of rectangular profile) are special cases of the given general solution.

NOMENCLATURE

a, b	u_1 coordinate dimensions
c, d	u_2 coordinate dimensions
e, f	u_3 coordinate dimensions
Bi, Bi_e	Biot numbers for fin
$\sqrt{g_1}, \sqrt{g_2}, \sqrt{g_3}$	metric coefficients
$\sqrt{g}$	Jacobian, $\sqrt{g} = \sqrt{g_1}\sqrt{g_2}\sqrt{g_3}$
h, h_e	convective coefficients, $W/m^2 \cdot K$
k	thermal conductivity, $W/m \cdot K$
Q	heat transfer rate, W
Q_{cond}, Q_{fin}	conduction heat transfer rate, W
Q_{conv}	convection heat transfer rate, W
R_{fin}	thermal resistance of fin, K/W
T	temperature, K
t	thickness, m
r, ϕ, z	circular cylinder coordinates
r, ϕ, ψ	spherical coordinates
u_1, u_2, u_3	curvilinear coordinates

Greek Symbols

Φ	fin function
θ	temperature excess $\theta = T - T_{\text{f}}$, K

Subscripts

b	fin base
e	fin tip
f	fluid

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INTRODUCTION

Many conduction problems in practice can be modeled as extended surfaces because conduction frequently occurs in thin metallic solids which are cooled by gases, e.g. air. Since the heat transfer takes place along one coordinate direction and the temperature variation through the thickness (perpendicular to the flow direction) is negligible, the temperature is essentially a function of one space coordinate only provided the appropriate coordinate system is used in the problem formulation.

Steady, one-dimensional conduction with convection cooling solutions are available for several simple geometries such as longitudinal extended surfaces having different profiles such as rectangular, triangular, concave and convex parabolic. Solutions for heat transfer through pin fins having profiles as described above are also of some interest to thermal analysts. Another very important case corresponds to heat transfer through extended surfaces which are part of or attached to circular tubes. The complete solutions for the three families of geometries are available in the excellent text of Kern and Kraus[1].

Although these geometries are useful for many conventional applications, there are many other cases for which these solutions are not useful and their application may lead to large errors in estimating the total heat transfer rate through the system.

There is, therefore, a need to develop equations and solutions for other geometries not covered by Kern and Kraus[1], and which are not available in the open literature.

A general equation will be developed in orthogonal curvilinear coordinates. The developed general equation will reduce to the particular cases found in Kern and Kraus[1] and Carslaw and Jaeger[2]. There are many other cases which arise from the general equation, e.g., solutions which are valid for conduction through cylindrical and spherical shells, to name only two.

GENERAL PROBLEM FORMULATION

Governing Differential Equations

The general equation can be obtained in a direct manner. It is assumed that one can select a coordinate system which is compatible with the geometry of the physical system. A proper choice allows one to assume that the temperature field is one-dimensional, i.e. $T(u_1)$ where u_1 is one of the three curvilinear coordinates. The conduction takes place in a thin curvilinear shell $c \le u_2 \le d$ of thickness $t = d - c$ whose surfaces $u_2 = c$ and $u_2 = d$ are convectively cooled through two different film coefficients h_1 and h_2 in the general case. For many applications one can assume that the film coefficients are equal. There is no conduction along the u_3-coordinate which means that constant u_3 surfaces are adiabatic. The above assumption that $T(u_1)$ is valid provided the fin Biot numbers are sufficiently small, i.e. $Bi_1 = h_1 t/k < 0.1$ and $Bi_2 = h_2 t/k < 0.1$.

Applying the Fourier rate equation to an appropriate differential control volume of sides $ds_1 = \sqrt{g_1}\, du_1, ds_2 = \sqrt{g_2}\, du_2, ds_3 = \sqrt{g_3}\, du_3$, and differential volume $dV = \sqrt{g}\, du_1 du_2 du_3$ as shown in Fig. 1, the heat conduction rate into the control volume is

$$\begin{aligned} dQ_{u_1} &= -kdA_1 \frac{d\theta}{ds_1} = -kds_2 ds_3 \frac{d\theta}{ds_1} \\ &= -k \frac{1}{\sqrt{g_1}} \frac{d\theta}{du_1} \sqrt{g_2}\sqrt{g_3}\, du_2\, du_3 = 0 \end{aligned} \quad (1)$$

where $\theta(u_1) = T(u_1) - T_f$ is the local temperature excess, and the metric coefficients $\sqrt{g_i}$ $(i = 1, 2, 3)$ are given in Moon and Spencer[3] for several orthogonal curvilinear coordinate systems.

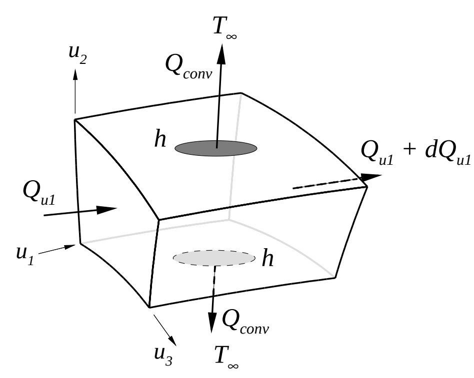

Fig. 1 General Curvilinear Fin Element

The heat convected from the differential volume is described by Newton's Law of Cooling which has the general form

$$dQ_{\text{conv}} = h\,\theta \sqrt{g_1}\sqrt{g_3}\, du_1 du_3 \quad (2)$$

Since both u_2−surfaces are convectively cooled through different film coefficients, the total local convection cooling is given by

$$dQ_{\text{conv}} = \left(h_1\,\theta \frac{\sqrt{g}}{\sqrt{g_2}}\, du_1 du_3 \right)_{u_2=c} + \left(h_2\,\theta \frac{\sqrt{g}}{\sqrt{g_2}} du_1\, du_3 \right)_{u_2=d} \tag{3}$$

If the heat transfer coefficients over the two convection surfaces $u_2 = c$ and $u_2 = d$ are equal, i.e. $h_1 = h_2 = h$, then the total convection loss from the differential control volume is given by the following relationship:

$$dQ_{\text{conv}} = \left[\left(\frac{\sqrt{g}}{\sqrt{g_2}} \right)_{u_2=c} + \left(\frac{\sqrt{g}}{\sqrt{g_2}} \right)_{u_2=d} \right] h\,\theta\, du_1\, du_3 \tag{4}$$

The assumption of equal convection heat transfer coefficients is not too restrictive.

A heat balance over the control surfaces of the differential control volume is given by:

$$dQ_{u_1} = dQ_{\text{conv}} + dQ_{u_1+du_1} \tag{5}$$

After substitution of the previously developed conduction and convection relationships into the heat balance, and integration over the cross-section of the fin, one obtains the differential equation for a curvilinear extended surface or fin:

$$\frac{d}{du_1} \left[\frac{d\theta}{du_1} \int_{u_3} \int_{u_2} \frac{\sqrt{g}}{g_1}\, du_2\, du_3 \right] - \frac{h}{k}\,\theta \int_{u_3} \left[\left(\frac{\sqrt{g}}{\sqrt{g_2}} \right)_{u_2=c} + \left(\frac{\sqrt{g}}{\sqrt{g_2}} \right)_{u_2=d} \right] du_3 = 0 \tag{6}$$

The above differential equation is applicable to isotropic fins whose surfaces u_2 = constant are convectively cooled through equal and constant film coefficients. The u_3−surfaces are adiabatic.

Boundary Conditions

Boundary conditions at the fin base ($u_1 = a$) and the fin tip ($u_1 = b$) are required to complete the mathematical description of the problem.

The conventional boundary condition at the base is perfect contact:

$$\theta = \theta_b, \qquad u_1 = a \tag{7}$$

where $\theta_b = T(u_1 = a) - T_f$. The general convection boundary condition is specified at the fin tip provided the fin is not truncated:

$$\frac{d\theta}{du_1} = -\frac{h_e}{k}\sqrt{g_1}\,\theta, \qquad u_1 = b \tag{8}$$

The film coefficient h_e at the fin tip is chosen to be different from the value along the sides of the fin. The temperature distribution θ, which is the solution of Eq.(6), will be reported in subsequent sections.

Fin Heat Flow Rate and Fin Resistance

The heat flow rate through the fin is obtained from the following general expression:

$$Q_{\text{fin}} = \left[\int_{u_3} \int_{u_2} -k \frac{\sqrt{g}}{g_1} \frac{d\theta}{du_1}\, du_2\, du_3 \right]_{u_1=a} \tag{9}$$

which is the conduction through the fin base. Another useful parameter, fin resistance, defined as

$$R_{\text{fin}} = \frac{\theta_b}{Q_{\text{fin}}} \tag{10}$$

will be presented in subsequent sections.

In the following sections the general fin equation will be applied to several coordinate systems to illustrate its utility. After the conventional solutions are presented, a few special cases will be considered. These are i) cylindrical shell fin, ii) spherical shell fin, iii) a wedge fin, and the iv) conical fin. Temperature distributions, heat flow rates and fin resistances will be presented for these examples.

CARTESIAN COORDINATES

The first example is based on the Cartesian coordinates (x, y, z) where, for convenience, we select the curvilinear coordinates u_1, u_2, u_3 to correspond to $u_1 = x, u_2 = y, u_3 = z$. The metric coefficients are therefore $\sqrt{g_1} = 1, \sqrt{g_2} = 1, \sqrt{g_3} = 1$ and $\sqrt{g} = 1$. The conduction occurs along the x−axis and convection occurs at the $y = c = 0$ and $y = d = t$ surfaces. The thickness of the fin is taken to be t. The width of the longitudinal fin of rectangular profile is w and the ranges of the coordinates are $0 \le x \le L$, $0 \le y \le t$, $0 \le z \le w$, as shown in Fig. 2. For this case, the general fin equation reduces to

$$\frac{d}{dx} \left[\frac{d\,\theta}{dx} \int_{z=0}^{z=w} \int_{y=0}^{y=t} dydz \right] - \frac{2h}{k}\,\theta \left[\int_{z=0}^{z=w} dz \right] = 0 \tag{11}$$

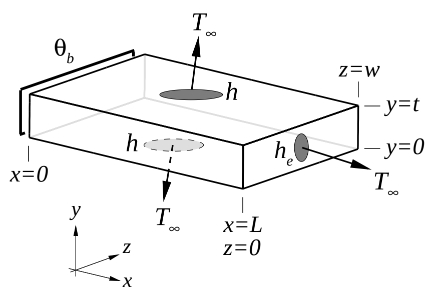

Fig. 2 Longitudinal Fin of Rectangular Profile

After integration and some simplifications, and with the introduction of the fin parameter $m^2 = 2h/(kt)$ we obtain the well-known fin equation:

$$\frac{d^2\theta}{dx^2} - m^2\,\theta = 0 \qquad 0 \le x \le L \tag{12}$$

whose solution in hyperbolic form is

$$\theta = C_1 \cosh(mx) + C_2 \sinh(mx) \tag{13}$$

Applying the two boundary conditions:

$$\theta(0) = \theta_b, \qquad \frac{d\theta(L)}{dx} = -\frac{h_e}{k}\,\theta(L) \tag{14}$$

gives the general solution

$$\frac{\theta}{\theta_b} = \cosh(mx) - \Phi\,\sinh(mx) \tag{15}$$

where the auxiliary function Φ is defined as

$$\Phi = \frac{mL \tanh(mL) + Bi_e}{mL + Bi_e \tanh(mL)} \tag{16}$$

The fin tip parameter is $Bi_e = h_e L/k$. The auxiliary function becomes $\Phi = \tanh(mL)$ if the fin tip is insulated ($Bi_e = 0$), and it becomes $\Phi = \coth(mL)$ if there is no resistance to heat transfer at the fin tip ($Bi_e = \infty$).

For the general case the heat flow rate through the fin base is obtained from

$$Q_{\text{fin}} = k\,w\,t\,m\,\Phi\,\theta_b \tag{17}$$

and the thermal resistance of the longitudinal fin is given by

$$R_{\text{fin}} = \frac{1}{k\,w\,t\,m\,\Phi} \tag{18}$$

Insulated Fin Tip

If the fin tip $x = L$ is insulated, $Bi_e = 0$, the temperature distribution reduces to

$$\frac{\theta}{\theta_b} = \cosh mx - \tanh mL\ \sinh mx \tag{19}$$

Perfect Contact at Fin Tip

If the fin tip is in perfect contact with the coolant, the temperature distribution becomes

$$\frac{\theta}{\theta_b} = \cosh mx - \coth mL\ \sinh mx \tag{20}$$

CIRCULAR CYLINDER COORDINATES

There are three fin equations for conduction in circular cylindrical shells. They correspond to conduction (i) in the radial r-direction, (ii) conduction in the angular ψ-direction, and (iii) conduction in the axial z-direction. In each case cited, convection losses take place through particular bounding surfaces.

Radial Conduction with Convection from $z-$ Surfaces

For completeness and to further illustrate the utility of the general formulation, the radial conduction case will be considered first. Here we select the curvilinear coordinates to be: $u_1 = r, u_2 = z, u_3 = \psi$; therefore we have $\sqrt{g_1} = 1, \sqrt{g_2} = 1, \sqrt{g_3} = r, \sqrt{g} = r$ The ranges of the coordinates are: $a \le r \le b,\ 0 \le \psi \le 2\pi,\ 0 \le z \le t$ where t is the thickness of the circular annular fin, as shown in Fig. 3. The two surfaces $z = 0$ and $z = t$ are convectively cooled through uniform film coefficients h.

The general fin equation becomes after substitution of the above parameters and ranges of the coordinates:

$$\frac{d}{dr}\left[\frac{d\theta}{dr}\int_0^t \int_0^{2\pi} r\,d\psi\,dz\right] - \frac{2h\theta}{k}\int_0^{2\pi} r\,d\psi = 0 \tag{21}$$

After the integrations and the introduction of the fin parameter $m^2 = 2h/(kt)$, it can be written in the conventional form:

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} - m^2\theta = 0, \quad a \le r \le b \tag{22}$$

which is the modified Bessel equation of order zero. Its solution consists of the modified Bessel functions $I_0(\cdot), K_0(\cdot)$, and it is written as

$$\theta = C_1 I_0(mr) + C_2 K_0(mr) \tag{23}$$

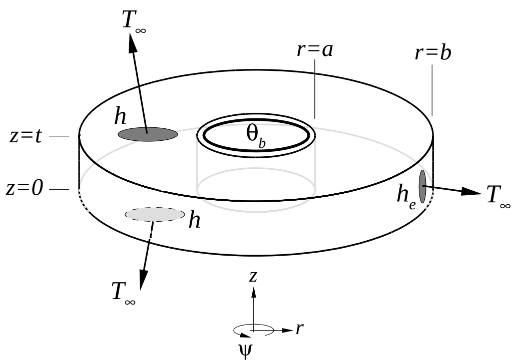

Fig. 3 Circular Annular Fin

The two constants of integration are obtained from the boundary conditions:

$$\theta(a) = \theta_b \quad \text{and} \quad \frac{d\theta(b)}{dr} = -\frac{h_e}{k}\theta(b) \tag{24}$$

Application of the boundary conditions gives the following relationships between the constants of integration:

$$C_1 = \frac{\theta_b}{I_0(ma) + \Phi\, K_0(ma)} \quad \text{and} \quad C_2 = -\Phi\, C_1 \tag{25}$$

The fin function Φ is given by

$$\Phi = \frac{mbI_1(mb) + Bi_e I_0(mb)}{mbK_1(mb) - Bi_e K_0(mb)} \tag{26}$$

where the parameter $Bi_e = \dfrac{h_e b}{k}$ accounts for the end cooling, and $I_1(mb), K_1(mb)$ are modified Bessel functions. The temperature distribution is given by

$$\frac{\theta(r)}{\theta_b} = \frac{I_0(mr) + \Phi\, K_0(mr)}{I_0(ma) + \Phi\, K_0(ma)}, \quad a \le r \le b \tag{27}$$

The heat flow rate through the fin is obtained by the application of Fourier's rate equation at the fin base:

$$Q_{\text{fin}} = \int_0^t \int_0^{2\pi} -kr\frac{d\theta}{dr} d\psi\, dz \quad \text{at} \quad r = a \tag{28}$$

Substitution of the temperature distribution and evaluation gives:

$$Q_{\text{fin}} = 2\pi\sqrt{2ha^2kt}\;\theta_b \left[\frac{\Phi K_1(ma) - I_1(ma)}{\Phi K_0(ma) + I_0(ma)}\right] \tag{29}$$

The fin resistance is obtained directly through Eq.(10):

$$R_{\text{fin}} = \frac{\theta_b}{Q_{\text{fin}}} = \frac{1}{2\pi\sqrt{2ha^2kt}} \left[\frac{\Phi K_0(ma) + I_0(ma)}{\Phi K_1(ma) - I_1(ma)}\right] \tag{30}$$

The results presented above are in agreement with those given by Kern and Kraus[1] and Yovanovich et al[4]. Two special cases arise from the general solution: (a) for the insulated fin tip and (b) for the fin with perfect contact at the fin tip.

Insulated Fin Tip

For this case $Bi_e = 0$ and the general solution yields:

$$\Phi = \frac{I_1(mb)}{K_1(mb)} \tag{31}$$

and

$$\frac{\theta(r)}{\theta_b} = \left[\frac{I_0(mr)K_1(mb) + K_0(mr)I_1(mb)}{I_0(ma)K_1(mb) + K_0(ma)I_1(mb)}\right] \tag{32}$$

and

$$R_{\text{fin}} = \frac{1}{2\pi\sqrt{2ha^2kt}} \left[\frac{K_0(ma)I_1(mb) + I_0(ma)K_1(mb)}{K_1(ma)I_1(mb) - I_1(ma)K_1(mb)}\right] \tag{33}$$

Perfect Contact at the Fin Tip

For this case $Bi_e = \infty$ and the general solution yields:

$$\Phi = -\frac{I_0(mb)}{K_0(mb)} \tag{34}$$

and

$$\frac{\theta(r)}{\theta_b} = \left[\frac{I_0(mr)K_0(mb) - K_0(mr)I_0(mb)}{I_0(ma)K_0(mb) - K_0(ma)I_0(mb)}\right] \tag{35}$$

and

$$R_{\text{fin}} = \frac{1}{2\pi\sqrt{2ha^2kt}} \left[\frac{K_0(ma)I_0(mb) - I_0(ma)K_0(mb)}{K_1(ma)I_0(mb) + I_1(ma)K_0(mb)}\right] \tag{36}$$

Radial Conduction and Convection from the ψ-Surfaces

The wedge-shaped fin which is modeled in this section is shown in Fig. 4. It consists of the sector: $0 \le r \le b, 0 \le \psi \le \alpha, 0 \le z \le L$ which is convectively cooled at the $\psi = 0$ and $\psi = \alpha$ surfaces through uniform heat transfer coefficients. The Biot number $Bi = h b\alpha/(2k) < 0.1$. Substitution of the parameters into the general fin equation with appropriate limits gives:

$$\frac{d}{dr}\left[\frac{d\theta}{dr}\int_0^L \int_0^\alpha r d\psi dz\right] - \frac{2h}{k}\int_0^L dz = 0 \tag{37}$$

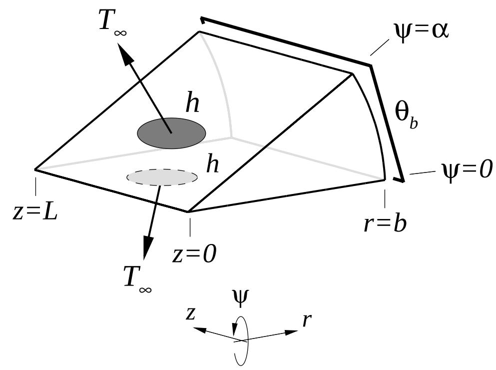

Fig. 4 Wedge Shaped Fin

which can be put into the conventional form of the fin equation after completion of the integrations and the introduction of the fin parameter $m^2 = \dfrac{2h}{\alpha k}$:

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} - m^2\frac{1}{r}\theta = 0 \qquad (38)$$

The above equation is a modified Bessel equation whose solution is

$$\theta(r) = C_1 I_0(2m\sqrt{r}) + C_2 K_0(2m\sqrt{r}) \qquad (39)$$

The boundary conditions are

$$\theta(0) \quad \text{is finite and} \quad \theta(b) = \theta_b \qquad (40)$$

The first boundary condition requires that the modified Bessel function $K_0(\cdot)$ should be eliminated from the solution; therefore we must put $C_2 = 0$ in the solution. The isothermal condition at $r = b$ requires

$$C_1 = \frac{\theta_b}{I_0(2m\sqrt{b})} \qquad (41)$$

With this value for the second constant of integration we obtain the temperature distribution:

$$\frac{\theta(r)}{\theta_b} = \frac{I_0(2m\sqrt{r})}{I_0(2m\sqrt{b})}, \qquad 0 \le r \le b \qquad (42)$$

The heat flow rate through this fin is obtained by the application of the Fourier rate equation at the boundary $r = b$:

$$Q_{\text{fin}} = kb\alpha L\frac{d\theta}{dr} = 2kL\alpha\sqrt{b}\theta_b\frac{I_1(2m\sqrt{b})}{I_0(2m\sqrt{b})} \qquad (43)$$

and the fin resistance is given by

$$R_{\text{fin}} = \frac{1}{2kL\alpha m\sqrt{b}}\frac{I_0(2m\sqrt{b})}{I_1(2m\sqrt{b})} \qquad (44)$$

Axial Conduction and Convection from $r-$ Surfaces

For the next example we consider the hollow circular cylinder of radii (a, b) and length L shown in Fig. 5. One end $z = 0$ of the cylindrical shell is maintained at $T = T_b$ while the other end $z = L$ is convectively cooled through a film coefficient h_e. The inner and outer boundaries $r = a$ and $r = b$ are convectively cooled by the same fluid through similar film coefficients h.

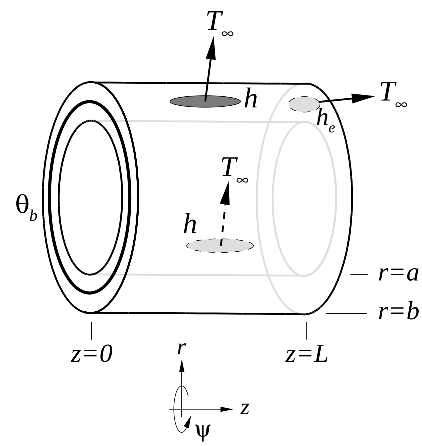

Fig. 5 Hollow Circular Cylinder

Here we select the curvilinear coordinates to be: $u_1 = z, u_2 = r, u_3 = \psi$; therefore we have $\sqrt{g_1} = 1, \sqrt{g_2} = 1, \sqrt{g_3} = r, \sqrt{g} = r$. The ranges of the coordinates are: $a \le r \le b$, $0 \le \psi \le 2\pi$, $0 \le z \le L$ and t $(=b-a)$ is the thickness of the shell. The condition $Bi = ht/k < 0.1$ must be satisfied.

Substitution into the general fin equation with appropriate limits of integration yields

$$\frac{d}{dz}\int_0^{2\pi}\int_a^b r\frac{d\theta}{dz}\,dr\,d\psi - \left[\int_0^{2\pi}\frac{h\theta a}{k}d\psi + \int_0^{2\pi}\frac{h\theta b}{k}d\psi\right] = 0 \qquad (45)$$

After integration one finds

$$\pi(b^2 - a^2)\frac{d^2\theta}{dz^2} - \frac{h\theta}{k}\left[2\pi(a+b)\right] = 0 \qquad (46)$$

The last equation can be put into the conventional form of the fin equation through the introduction of the fin parameter:

$$m^2 = \frac{2h}{k(b-a)} \qquad (47)$$

The fin equation becomes:

$$\frac{d^2\theta}{dz^2} - m^2\theta = 0, \qquad 0 \le z \le L \tag{48}$$

The form of this equation is identical to the fin equation given above under the Cartesian coordinates except for the replacement of x by z. Since the boundary conditions are identical, the solution of the fin equation is also identical. The results for the temperature distribution and the fin function Φ can be used directly provided the x variable is replaced by the z variable and the appropriate form of the fin parameter m is used. The heat flow rate through the fin is given by

$$Q = k\pi(b^2 - a^2)\ m\ \Phi\ \theta_b \tag{49}$$

and the fin resistance is given by

$$R = \frac{1}{k\pi(b^2 - a^2)\ m\ \Phi} \tag{50}$$

The limiting results for the fin function Φ for a fin with insulated fin tip and a fin with perfect contact at the fin tip are equally true for this cylindrical fin.

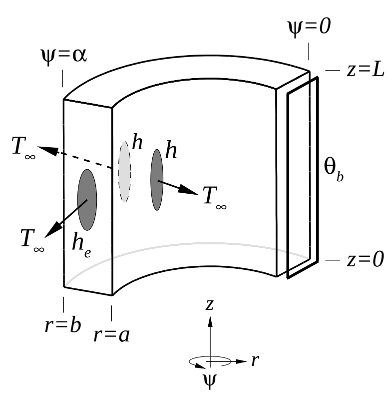

Fig. 6 Edge Heated Cylindrical Shell Section

Angular Conduction with Convection from $\psi-$ Surfaces

Here we consider the portion of the cylindrical shell of radii (a, b), length L which is heated along the edge $\psi = 0$ where $\theta = \theta_b$ and which is convectively cooled along the edge $\psi = \alpha$ through the uniform film coefficient h_e (see Fig. 6). The inner $(r = a)$ and outer $(r = b)$ boundaries are convectively cooled through equal film coefficients h. The ends of this cylindrical shell are adiabatic. For this fin the temperature excess is a function of the ψ coordinate. Here we select the curvilinear coordinates to be: $u_1 = \psi, u_2 = r, u_3 = z$; therefore we have $\sqrt{g_1} = r$, $\sqrt{g_2} = 1$, $\sqrt{g_3} = 1$, $\sqrt{g} = r$. The ranges of the coordinates are: $a \le r \le b$, $0 \le \psi \le \alpha$, $0 \le z \le L$. The fin parameter $Bi = h(b-a)/k < 0.1$.

Substitution of the above into the general fin equation gives:

$$\frac{d}{d\psi}\left[\int_0^L \int_a^b \frac{1}{r}\frac{d\theta}{d\psi}\, dr\, dz\right] - \left[\int_0^L \frac{h\theta a}{k} dz + \int_0^L \frac{h\theta b}{k} dz\right] = 0 \tag{51}$$

After completion of the integrations the last result can be put into the following form:

$$\frac{d^2\theta}{d\psi^2} - m^2\ \theta = 0, \quad 0 \le \psi \le \alpha \tag{52}$$

where the fin parameter has been defined as

$$m^2 = \frac{h(a+b)}{k\ln(\frac{b}{a})} \tag{53}$$

We note that this parameter is dimensionless. The solution of the above fin equation is

$$\theta(\psi) = C_1 \cosh m\psi + C_2 \sinh m\psi \tag{54}$$

Application of the boundary conditions to the above solution yields the two equations for the constants of integration:
at $\psi = 0$,

$$C_1 = \theta_b \tag{55}$$

and at $\psi = \alpha$,

$$-kL\ln(\frac{b}{a})\,[C_1 m \sinh m\alpha + C_2 m \cosh m\alpha] = h_e L(b-a)\,[C_1 \sinh m\alpha + C_2 \cosh m\alpha] \tag{56}$$

Solving for the constants of integration one obtains:

$$C_1 = \theta_b \quad \text{and} \quad C_2 = -\Phi\ C_1 \tag{57}$$

where the fin function is now defined as

$$\Phi = \frac{m \sinh m\alpha + (Bi_e/\ln(b/a)) \cosh m\alpha}{m \cosh m\alpha + (Bi_e/\ln(b/a)) \sinh m\alpha} \tag{58}$$

with $Bi_e = h(b-a)/k$. The temperature distribution is given by

$$\frac{\theta(\psi)}{\theta_b} = \cosh m\psi - \Phi\ \sinh m\psi, \quad 0 \le \psi \le \alpha \tag{59}$$

The fin heat flow rate is obtained by integration of the following equation over the base conduction area:

$$Q = \left[\int_0^L \int_a^b -k \frac{1}{r} \frac{d\theta}{d\psi} \, dr \, dz \right]_{\psi=0} \tag{60}$$

Therefore,

$$Q_{\text{fin}} = kmL \; \Phi \; \ln(\frac{b}{a}) \; \theta_b \tag{61}$$

The fin resistance is therefore given by

$$R_{\text{fin}} = \frac{1}{kmL \; \Phi \; \ln(\frac{b}{a})} \tag{62}$$

The general result reduces to simpler forms if the fin tip is insulated or if it is in perfect contact with the coolant. For the insulated tip case, $Bi_e = 0$, $\Phi = \tanh m\alpha$ and we have for the temperature distribution:

$$\frac{\theta(\psi)}{\theta_b} = \cosh m\psi - \tanh m\alpha \; \sinh m\psi, \quad 0 \le \psi \le \alpha \tag{63}$$

and for the heat flow rate we get

$$Q_{\text{fin}} = k \; m \; L \; \tanh(m\alpha) \; \ln(\frac{b}{a})\theta_b \tag{64}$$

and the fin resistance is given by

$$R_{\text{fin}} = \frac{1}{k \; m \; L \; \tanh(m\alpha) \; \ln(\frac{b}{a})} \tag{65}$$

For the perfect contact tip condition, $Bi_e = \infty$, and $\Phi = \coth m\alpha$. The temperature distribution, the fin heat flow rate and the fin resistance can be obtained directly from the above three results with $\tanh m\alpha$ replaced by $\coth m\alpha$.

SPHERICAL COORDINATES

There are three fin equations for conduction in spherical shells. They correspond to conduction (i) in the radial r-direction, (ii) conduction in the polar ϕ-direction, and (iii) conduction in the ψ-direction. In each case there are convection losses through bounding surfaces.

Radial Conduction with Convection from $\phi-$ Surfaces

The radial conduction case is considered first. The heat transfer occurs in a system which is bounded by the surfaces $r = a$ and $r = b$ and the conical surface $\phi = \alpha$ as shown in Fig. 7. The surface $r = a$ is maintained isothermal at temperature θ_b and the surface $r = b$ is convectively cooled through a uniform film coefficient h_e. The conical surface $\phi = \alpha$ is convectively cooled through a uniform film coefficient h. The Biot number is $Bi = hb \sin\alpha / k < 0.1$. The temperature is one-dimensional, $\theta(r)$, in this system.

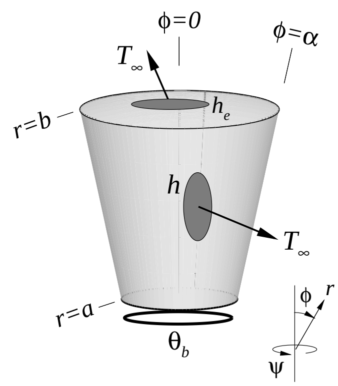

Fig. 7 Conical Section of a Spherical Shell

Here we select the curvilinear coordinates to be: $u_1 = r, u_2 = \phi, u_3 = \psi$; therefore we have $\sqrt{g_1} = 1, \sqrt{g_2} = r, \sqrt{g_3} = r\sin\phi, \sqrt{g} = r^2 \sin\phi$. The ranges of the coordinates are: $a \le r \le b$, $0 \le \phi \le \alpha$, $0 \le \psi \le 2\pi$.

The general fin equation becomes after substitution of the above parameters and ranges on the coordinates:

$$\frac{d}{dr}\left[\frac{d\theta}{dr}\int_0^{2\pi}\int_0^{\alpha} r^2 \sin\phi \, d\phi \, d\psi\right] - \frac{h\theta}{k}\int_0^{2\pi} r\sin\alpha d\psi = 0 \tag{66}$$

which after the integrations and the introduction of the fin parameter $m^2 = \dfrac{h\sin\alpha}{k}$ can be written in the conventional form

$$\frac{d^2\theta}{dr^2} + \frac{2}{r}\frac{d\theta}{dr} - m^2\frac{1}{r}\theta = 0, \quad a \le r \le b \tag{67}$$

which is the modified Bessel equation of order zero. Its solution consists of the modified Bessel functions $I_1(\cdot), K_1(\cdot)$, and it is written as

$$\theta = \frac{1}{\sqrt{r}}\left[C_1 I_1(2m\sqrt{r}) + C_2 K_1(2m\sqrt{r})\right] \tag{68}$$

The application of the boundary conditions given above yields the constants of integration:

$$C_1 = \frac{\theta_b\sqrt{a}}{I_1(2m\sqrt{a}) + \Phi K_1(2m\sqrt{a})} \tag{69}$$

and

$$C_2 = \Phi C_1 \tag{70}$$

with the fin function defined as

$$\Phi = \left[\frac{m\sqrt{b} I_2(2m\sqrt{b}) + Bi_e I_1(2m\sqrt{b})}{m\sqrt{b} K_2(2m\sqrt{b}) - Bi_e K_1(2m\sqrt{b})} \right] \tag{71}$$

where the end cooling is characterized by the parameter $Bi_e = h_e b/k$.

The temperature distribution is given by

$$\frac{\theta}{\theta_b} = \sqrt{\frac{a}{r}} \left[\frac{I_1(2m\sqrt{r}) + \Phi K_1(2m\sqrt{r})}{I_1(2m\sqrt{a}) + \Phi K_1(2m\sqrt{a})} \right] \tag{72}$$

The heat flow rate through the fin is obtained by the application of the Fourier rate equation at the fin base $r = a$:

$$Q_{\text{fin}} = \left[\int_0^{2\pi} \int_0^{\pi/2} -kr^2 \sin\phi \frac{d\theta}{dr} d\phi d\psi \right]_{r=a} \tag{73}$$

which becomes

$$Q_{\text{fin}} = 2\pi k m \sqrt{a}\, \theta_b \left[\frac{\Phi K_2(2m\sqrt{a}) - I_2(2m\sqrt{a})}{\Phi K_1(2m\sqrt{a}) + I_1(2m\sqrt{a})} \right] \tag{74}$$

The fin resistance is found to be

$$R_{\text{fin}} = \frac{1}{2\pi k m \sqrt{a}} \left[\frac{\Phi K_1(2m\sqrt{a}) + I_1(2m\sqrt{a})}{\Phi K_1(2m\sqrt{a}) - I_2(2m\sqrt{a})} \right] \tag{75}$$

The special cases which arise from the limits: $Bi_e = 0$ and $Bi_e = \infty$ are obtained from the fin function Φ which takes on the following two values respectively:

$$\Phi = \frac{I_2(2m\sqrt{b})}{K_2(2m\sqrt{b})} \tag{76}$$

and

$$\Phi = -\frac{I_1(2m\sqrt{b})}{K_1(2m\sqrt{b})} \tag{77}$$

These values are to be substitute into the expressions for $\theta(r)$, Q_{fin} and R_{fin}.

Conduction Along $\psi-$Coordinate with Convection from $r-$Surfaces

The spherical fin of interest is defined by the ranges: $a \le r \le b, \alpha \le \phi \le \pi/2, 0 \le \psi \le \pi$ as shown in Fig. 8. The edges $\phi = \alpha$ and $\phi = \pi/2$ are adiabatic. The edge $\psi = 0$ is maintained at temperature $\theta(0) = \theta_b$ while the other edge is convectively cooled through a uniform film coefficient h_e. The remaining two boundaries $r = a$ and $r = b$ are convectively cooled through uniform film coefficients h. Since the Biot number is $Bi = h(b-a)/(2k) < 0.1$, the temperature distribution is one-dimensional, $\theta = \theta(\psi)$.

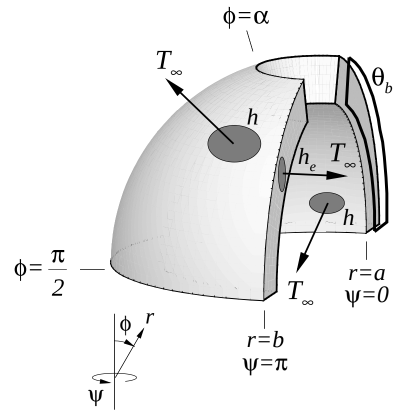

Fig. 8 Spherical Shell Fin

The orthogonal curvilinear coordinates are selected such that $u_1 = \psi, u_2 = r, u_3 = \phi$. Therefore we have $\sqrt{g_1} = r\sin\phi$, $\sqrt{g_2} = 1$, $\sqrt{g_3} = r$, $\sqrt{g} = r^2 \sin\phi$. Substitution into the general fin equation with appropriate limits gives:

$$\frac{d}{d\psi} \left[\frac{d\theta}{d\psi} \int_a^b \int_\alpha^\beta \frac{1}{\sin\phi} d\phi dr \right] - \left[\frac{h\theta}{k} \int_\alpha^\beta r^2 \sin\phi d\phi \right]_{r=a,b} = 0 \tag{78}$$

After completion of the integrations and the introduction of the fin parameter:

$$m^2 = \frac{h(a^2+b^2)(\cos\alpha - \cos\beta)}{k(b-a)\ln\left[\dfrac{\tan(\beta/2)}{\tan(\alpha/2)}\right]} \tag{79}$$

the general fin equation reduces to

$$\frac{d^2\theta}{d\psi^2} - m^2\theta = 0 \tag{80}$$

whose solution in hyperbolic form is

$$\theta = C_1 \cosh m\psi + C_2 \sinh m\psi \tag{81}$$

Solving for the constants of integration gives

$$C_1 = \theta_b \qquad \text{and} \qquad C_2 = -\Phi C_1 \tag{82}$$

with the fin function defined as

$$\Phi = \frac{m \sinh m\pi + Bi_e \zeta \cosh m\pi}{m \cosh m\pi + Bi_e \zeta \sinh m\pi} \tag{83}$$

where

$$Bi_e = \frac{h_e}{k}\frac{(b+a)}{2} \qquad \text{and} \qquad \zeta = \frac{(\beta - \alpha)}{\ln\left[\frac{\tan(\beta/2)}{\tan(\alpha/2)}\right]} \tag{84}$$

The temperature distribution, the heat flow rate and the fin resistance can be obtained from the general equations. They are

$$\frac{\theta(\psi)}{\theta_b} = \cosh m\psi - \Phi \sinh m\psi \tag{85}$$

and

$$Q_{\text{fin}} = \left[\int_\alpha^\beta \int_a^b -k \frac{1}{\sin\phi}\frac{d\theta}{d\psi} dr d\phi\right]_{\psi=0}$$

$$= km(b-a)\Phi\theta_b \ln\left[\frac{\tan(\frac{\beta}{2})}{\tan(\frac{\alpha}{2})}\right] \tag{86}$$

and from the definition of fin resistance we get

$$R_{\text{fin}} = \frac{1}{km(b-a)\Phi \ln\left[\frac{\tan(\frac{\beta}{2})}{\tan(\frac{\alpha}{2})}\right]} \tag{87}$$

The two special cases which arise from the general end condition are: (i) $Bi_e = 0$ for which the fin function becomes $\Phi = \tanh m\pi$ and (ii) $Bi_e = \infty$ for which the fin function becomes $\Phi = \coth m\pi$. The temperature distribution, the heat flow rate and the fin resistance expressions reduce to simpler forms for these special cases.

SUMMARY

A general equation in orthogonal curvilinear coordinates for heat transfer through fins was developed, and the general fin equation for perfect contact at the fin base and end cooling was given. The general expression for heat transfer rate through the fin was obtained, and the fin resistance was defined.

It was shown through several examples how the general equation reduces to the special cases considered in most heat transfer texts.

The temperature distribution, heat transfer rate and corresponding fin resistance were given for three coordinate systems: Cartesian, circular cylinder and spherical.

Solutions for (i) insulated fin end and (ii) perfect contact at the fin end were also presented. It was shown that the well-known solutions (longitudinal fin of rectangular profile, circular annular fin of rectangular profile) are special cases of the given general solution.

The general equations can be used to obtain results for many other special orthogonal coordinate systems such oblate and prolate spheroidal coordinates: bicylinder coordinates, elliptic cylinder coordinates and parabolic cylinder coordinates for example.

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REFERENCES

[1]Kern, D.Q., and Kraus, A.D., *Extended Surface Heat Transfer*, McGraw-Hill Book Company, 1972.

[2]Carslaw, H.S. and Jaeger, J.C., *Conduction of Heat in Solids*, 2nd edition, Oxford University Press, 1959.

[3]Moon, P. and Spencer, D.E., *Field Theory Handbook*, 2nd edition, Springer-Verlag, New York, 1971.

[4]Yovanovich, M.M., Culham, J.R., and Lemczyk, T.F., *Simplified Solutions to Circular Annular Fins with Contact Resistance and End Cooling*, J. Thermophysics and Heat Transfer, Vol. 2, No. 2, pp. 152-157, April, 1988.