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# Analytical Method for Forced Convection from Flat Plates, Circular Cylinders, and Spheres

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**A simple general method is developed to predict forced convection heat transfer from isothermal body shapes such as flat plates, infinite circular cylinders, and spheres for a wide range of both Reynolds number and Prandtl number. The proposed method is based on linearization of the thermal energy equation that is accomplished by the introduction of an effective velocity that is related to the freestream velocity. Next, the linear energy equation is transformed to an equivalent transient heat conduction equation that has existing solutions. These solutions are retransformed to the final expression as a function of the effective velocity that is defined in the limits of $Pr \rightarrow \infty$ and $Pr \rightarrow 0$ using scaling analysis. The approximate analytic solutions are in closed form, they are simple and quite accurate when compared with previous experimental and analytical studies.**

## Nomenclature

| | |
|---|---|
| $A$ | = surface area, m$^2$ |
| $C_D$ | = constant in Table 1 |
| $C_i$ | = constant in **Eq.** (2) |
| $C_0$ | = constant in **Eq.** (16) |
| $D$ | = sphere or cylinder diameter, m |
| $F(Pr, \gamma_i)$ | = function defined in Eq. (3) |
| $Fo_i$ | = Fourier number, $\alpha t/\mathscr{L}^2$ |
| $h$ | = coefficient of convection heat transfer, W/m$^2$K |
| $\hbar_1, \hbar_2, \hbar_3$ | = curvilinear scale factors |
| $i$ | = notation index |
| $ierfc$ | = first integral of complementary error function |
| $i^2erfc$ | = second integral of complementary error function |
| $k$ | = thermal conductivity, W/m K |
| $L$ | = flat plate and cylinder length, m |
| $\mathscr{L}$ | = arbitrary scale length, m |
| $Nu_i$ | = area-averaged Nusselt number, $\mathscr{L}h/k$ |
| $Nu_i^0$ | = area-averaged Nusselt number at the diffusive limit |
| $Pe_i$ | = Peclet number, $\mathscr{L}V_\infty/\alpha$ |
| $Pr$ | = Prandtl number, $\nu/\alpha$ |
| $Q$ | = total heat flow rate, W |
| $q$ | = heat flux, W/m$^2$ |
| $q_S$ | = heat flux at the surface, W/m$^2$ |
| $Re_i$ | = Reynolds number, $\mathscr{L}V_\infty/\nu$ |
| $Re_i(\theta)$ | = local Reynolds number, $\mathscr{L}V(\theta)/\nu$ |
| $r, \theta, Z$ | = cylindrical coordinates |
| $r, \theta, \phi$ | = spherical coordinates |
| $T$ | = temperature, K |
| $\boldsymbol{T}$ | = nondimensional temperature, $(T - T_\infty)/(T_S - T_\infty)$ |
| | = surface temperature, K |
| | = ambient temperature, K |
| $t$ | = time, s |
| $u_1, u_2, u_3$ | = velocities in curvilinear coordinates, m/s |
| $V$ | = local velocity at the edge of thermal boundary layer, m/s |
| $V(\theta)$ | = local velocity at the edge of hydrodynamic boundary layer, m/s |
| $V_\infty$ | = freestream velocity, m/s |
| $\bar{v}_e$ | = area-averaged effective velocity, m/s |
| $\bar{v}_e^\infty$ | = area-averaged effective velocity as $Pr \rightarrow \infty$, m/s |
| $\bar{v}_e^0$ | = area-averaged effective velocity as $Pr \rightarrow 0$, m/s |
| $\bar{v}_e^\infty(\theta)$ | = local effective velocity as $Pr \rightarrow \infty$, m/s |
| $W$ | = flat plate width, m |
| $X, Y, Z$ | = Cartesian coordinates |
| $x_1, x_2, x_3$ | = curvilinear coordinates |
| $\alpha$ | = thermal diffusivity, $k/C_p\rho$, m$^2$/s |
| $\gamma$ | = parameter in Eq. (18) |
| $\delta$ | = local thickness of hydrodynamic boundary layer, m |
| $\delta_{i2}$ | = Kronecker delta in $x_2$ direction |
| $\delta_T$ | = local thickness of thermal boundary layer, m |
| $\delta_D^T$ | = displacement thickness of thermal boundary layer, m |
| $\delta_M^T$ | = momentum thickness of thermal boundary layer, m |
| $\eta$ | = nondimensional quantity, $y/\delta$ |
| $\nu$ | = kinematic viscosity, m$^2$/s |
| $\rho$ | = mass density, kg/m$^3$ |

## Introduction

EXTERNAL forced convection heat transfer from isothermal or isoflux external convex surfaces is an important problem for engineers. There are many engineering systems that are modeled using forced convection, such as electronic components on printed circuit boards placed in cabinets, hot wire anemometer, and heat exchanger design.

A schematic of the different body shapes, which are investigated in this study, is shown in Fig. 1. These different body shapes are maintained at $T_S$ and the environment is maintained at $T_\infty$. These bodies are subjected to a uniform steady flow.

The present study is concerned with the effects of Reynolds number, Prandtl number, and velocity profiles on forced convection heat transfer from isothermal surfaces such as flat plates (FP), infinite circular cylinders (ICC) and spheres.

## Literature Survey

The objective of this section is to present a brief summary of some previous studies dealing with external forced con-

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**Table 1 Recalculation of the Hilpert[10] air data by Morgan,[9] $Nu_D = C_D Re_D^m$**

| $Re_D$ | $L/D$ | Hilpert[10] | | Morgan[9] | |
|---|---|---|---|---|---|
| | | $C_D$ | $m$ | $C_D$ | $m$ |
| 1–4 | 5,120 | 0.891 | 0.33 | — | — |
| 4–40 | 1,625–5,120 | 0.821 | 0.385 | 0.795 | 0.384 |
| 40–4,000 | 20–3,170 | 0.615 | 0.466 | 0.583 | 0.471 |
| 4,000–40,000 | 5.6–20 | 0.174 | 0.618 | 0.148 | 0.633 |
| 40,000–400,000 | 0.9–11.4 | 0.0239 | 0.805 | 0.0208 | 0.814 |

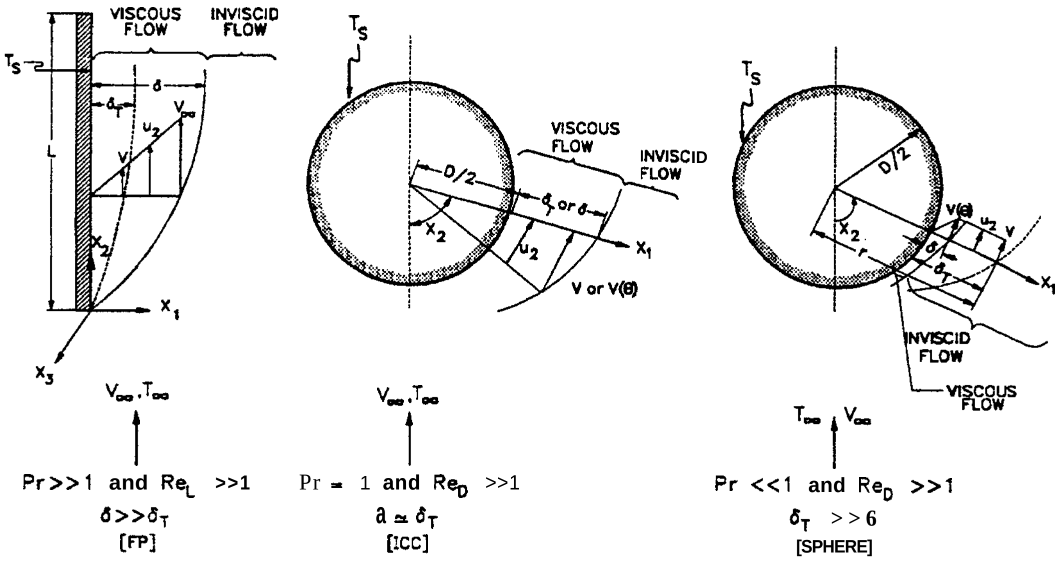


**Fig. 1 Schematic diagram of boundary layers over different body shapes.**

vection heat transfer from flat plates, infinite circular cylinders, and spheres. This summary will also show limitations in the previous investigations and their discrepancies.

## Flat Plates

Forced convection from isothermal flat plates has been investigated since the beginning of this century and different solutions have been developed for specific ranges of $Pr$ and $Re_L$. In addition, other investigations have been done to provide better understanding of the problem such as the recent study of Yovanovich et al.[1] However, the effect of velocity profiles as a function of Reynolds number on the heat transfer results has not been investigated in previous studies such as Pohlhausen (see Kays and Crawford[2]), Schlichting,[3] and Yovanovich et al.[1] Furthermore, there are disagreements between experimental studies and analytical studies such as the investigations of Slegel and Hawkins,[4] Parmalee and Huebscher,[5] and Knudsen and Katz.[6]

## Circular Cylinders

Numerous studies have been done to correlate heat transfer by forced convection from circular cylinders in crossflow in the following form: $Nu_\mathcal{D} = Nu_\mathcal{D}(Re_\mathcal{D}, Pr)$. However, all of them have failed to establish a general form: $Nu_D = Nu_\mathcal{D}(L/D, Re_D, Pr)$. In addition, there are many two-dimensional numerical studies,[7] however, the $L/D$ is not defined. Furthermore, Churchill and Bernstein[8] developed a general correlation for forced convection heat transfer from horizontal circular cylinders to cover a wide range of Reynolds number:

$$Nu_D = 0.3 + \left\{ 0.62 Re_D^{0.5} \frac{Pr^{1/3}}{[1 + (0.4/Pr)^{2/3}]} \cdot [1 + (Re_D/282{,}000)^{5/8}]^{4/5} \right\} \qquad (1)$$

for $10^2 \leq Re_D \leq 10^7$.

Equation (1) has been developed for infinite circular cylinder, i.e., $L >> D$ based on fitting many experimental data from the literature.

Morgan[9] presented a comprehensive literature review for this problem and developed various correlations for the crossflow over cylinders based on the experimental data of Hilpert[10] as shown in Table 1. One observes from Table 1 that the Morgan correlations are valid for specific values of $L/D$ and also specific ranges of Reynolds number.

## Spheres

Steady forced convection heat transfer from isothermal spheres into a substantial amount of fluid has been investigated experimentally, theoretically, and numerically by many researchers.[11]

The effects of $Pr$, $Re_D$ and velocity profiles on forced convection heat transfer have also been investigated for this case by Refai Ahmed and Yovanovich[12] using an approximate analytical method.

## Remarks

Exact solutions for forced convection heat transfer from arbitrary body shapes are very difficult to obtain, since the governing equations dealing with this type of problem are nonlinear. Furthermore, the researchers, who investigated this problem analytically, developed their solutions for specific ranges of Prandtl and Reynolds numbers. The difficulties of predicting forced convection heat transfer are summarized as follows:

1) Starting with the full governing equations creates difficulties to the analytical solution due to the nonlinear equations and the pressure change over arbitrary body shapes. In addition, consideration of flow separation over the body shapes precludes any further analysis of the governing equations.

2) Numerical techniques are expensive and also require large computational effort.

3) Experimental techniques are time-consuming, expensive, and test specific.

## Objectives

In the present investigation an approximate analytical study will be conducted to determine the effect of various parameters upon the area-averaged Nusselt number. One of the important goals of the present study is to consider the effects of Reynolds number, Prandtl number, and velocity profiles on the area-averaged Nusselt number. Also, a general model for forced convection heat transfer from flat plates, infinite circular cylinders, and spheres will be developed in the following form:

$$Nu_{\mathscr{I}} = Nu^0_{\mathscr{I}} + C_{\mathscr{I}}\sqrt{Re_{\mathscr{I}}}F(Pr, \gamma_{\mathscr{I}}) \tag{2}$$

where $Nu^0_{\mathscr{I}}$ is the area-averaged Nusselt number at the diffusive limit, $C_{\mathscr{I}}$ is a constant that depends on the body shape, and $F(Pr, \gamma_{\mathscr{I}})$ is a function that depends on fluid properties and velocity profiles through the parameter $\gamma_{\mathscr{I}}$, where $y$, is a function of Reynolds number.

### Theoretical Analysis

The proposed approximate analytical method can be outlined as follows: one must convert the energy equation to a linear partial differential equation; furthermore, the new linear partial differential equation can be transformed to a transient equation in order to obtain the final form of $Nu$, as a function of area-mean effective velocity $\bar{v}_e$; the next step is to examine the governing equations using scaling analysis for $Re_{\mathscr{I}} >> 1$ and $Pr >> 1$, and $Re$, $>> 1$ and $Pr << 1$ in order to determine $\bar{v}_e$; this is accomplished by applying scaling analysis inside the hydrodynamic boundary layer for both the continuity and momentum equations to obtain $\delta/\mathscr{L}$; in addition, one must apply scaling analysis inside the thermal boundary layer for both the continuity and energy equations in order to obtain $\delta_T/\mathscr{L}$; one can define $\bar{v}^{\infty}_e$ through momentum balances inside the thermal boundary layer; by contrast, one can obtain from the right side of the flow chart the definition of $\bar{v}^0_e$, which presents the case of $Pr << 1$ and $Re_{\mathscr{I}} >> 1$, by applying boundary-layer concepts; and finally, one applies a "blending" method in order to find $\bar{v}_e$ for all Prandtl numbers and then substitute it in the Nusselt number to obtain the final expression.

Figure 1 shows an isothermal FP, ICC, and sphere of temperature $T_S$ and length $L$ for the FP, and diameter $D$ for both the ICC and the sphere, which are immersed in a steady, laminar, and incompressible flow of a constant property fluid ($0 < Pr < \infty$) at constant temperature $T_\infty$ and uniform upstream velocity $V_\infty$. In addition, Fig. 1 describes the hydrodynamic and thermal boundary layers for various Prandtl numbers at $Re$, $>> 1$. The energy equation with negligible heat dissipation in orthogonal curvilinear coordinates is

$$\left[\frac{u_1}{\hbar_1}\frac{\partial T}{\partial x_1} + \frac{u_2}{\hbar_2}\frac{\partial T}{\partial x_2} + \frac{u_3}{\hbar_3}\frac{\partial T}{\partial x_3}\right] = \frac{\alpha}{\hbar_1\hbar_2\hbar_3}\left[\frac{\partial}{\partial x_1}\left(\frac{\hbar_2\hbar_3}{\hbar_1}\frac{\partial T}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{\hbar_1\hbar_3}{\hbar_2}\frac{\partial T}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{\hbar_1\hbar_2}{\hbar_3}\frac{\partial T}{\partial x_3}\right)\right] \tag{3}$$

where the parameters (scale factors, coordinates, and velocity components) for the FP, ICC, and sphere are given in Table 2. We further assume that viscous effects are confined to a thin hydrodynamic boundary layer $\delta$, and that the flow is inviscid outside the boundary layer. One can neglect conduction in the $x_2$ direction by comparing it with conduction in the $x_1$ direction. Table 3 shows the comparison between the order of magnitude of these terms that have been evaluated by scaling analysis.

**Table 2 Equivalent terms in curvilinear coordinates**

| | Scale factors | | | Coordinates | | | Velocity components | | |
|---|---|---|---|---|---|---|---|---|---|
| System | $\hbar_1$ | $\hbar_2$ | $\hbar_3$ | $x_1$ | $x_2$ | $x_3$ | $u_1$ | $u_2$ | $u_3$ |
| Cartesian | 1 | 1 | 1 | $X$ | $Y$ | $Z$ | $u$ | 71 | $w$ |
| Cylindrical | 1 | $\gamma$ | 1 | $r$ | $\theta$ | $z$ | 71, | $v_\theta$ | $v_z$ |
| Spherical | 1 | $r$ | $\gamma \sin\theta$ | $r$ | $\theta$ | $\phi$ | $v,$ | $v_\theta$ | $v_\phi$ |

**Table 3 Scaling analysis of the diffusive term for different body shapes**

| Body shape | $\frac{I}{\hbar_1\hbar_2\hbar_3}\frac{a}{\partial x_1}\left(\frac{\hbar_3\hbar_3}{\hbar_1}\frac{\partial T}{ax,)}\right.$ | $\frac{1}{\hbar_1\hbar_2\hbar_3}\frac{\partial}{\partial x_2}\left(\frac{\hbar_2\hbar_3}{\hbar_2}\frac{\partial T}{\partial x_2}\right)$ |
|---|---|---|
| FP | $1/\delta^2_T$ | $1/L^2$ |
| ICC | $1/\delta^2_T$ | 4/( |
| Sphere | $1/\delta^2_T$ | $4/(\theta D)^{2a}$ |

[a] $\theta > 0$.

Furthermore, heat diffusion and convection in the $x_3$ direction can be neglected for the following reasons: the variations of the temperature along the X? direction for the FP and ICC are negligible; and assuming axisymmetry, one can neglect the temperature variation in the $x_3$ direction for the sphere.

The advection terms on the left side of Eq. (3) are approximated by a single equivalent term, i.e., $(\bar{v}_e/\hbar_2)(\partial T/\partial x_2)$, where $\bar{v}_e$ is the area-average effective velocity that will be determined later. This idea has been proposed by Oseen to linearize the inertia term for creeping flow, where Oseen assumed the convective term to be $V_\infty \nabla \cdot u_i$ (for more details see Happel and Brenner[12]). In addition, the effective velocity has been introduced by Yovanovich et al.[1] and Jafarpur[13]; therefore, Eq. (3) becomes

$$\frac{\bar{v}_e}{\hbar_2}\frac{\partial T}{\partial x_2} = \frac{\alpha}{\hbar_1\hbar_2\hbar_3}\frac{\partial}{\partial x_1}\left(\frac{\hbar_2\hbar_3}{\hbar_1}\frac{\partial T}{\partial x_1}\right) \tag{4}$$

This equation is limited to the range $0 < x_2$, for the FP, and $0 \le x_2 \le \pi$, for the ICC and the sphere.

Equation (4) must be transformed to a transient heat conduction in order to find a suitable solution. Let us assume that the flow particles are moving with a constant velocity $\bar{v}_e$ around the body. Therefore, the particles will take time At to travel a distant $\hbar_2 x_2$. Furthermore, for $\Delta x_2 \to 0$ and At $\to$ 0. one can define

$$\bar{v}_e = \hbar_2 \frac{\partial x_2}{\partial t} \tag{5}$$

This concept was also used by Sideman[14] and Yovanovich et al.[1] Therefore, the energy equation can be transformed to the form of the transient heat conduction equation. Thus, by substituting Eq. (5) in Eq. (6), the energy equation can be written in the general form

$$\frac{\partial T^*}{\partial t} = \frac{\alpha}{\hbar_1\hbar_2\hbar_3}\left[\frac{\partial}{\partial x_1}\left(\frac{\hbar_2\hbar_3}{\hbar_1}\frac{\partial T^*}{\partial x_1}\right)\right] \tag{6}$$

where

$$x_1 \ge 0 \quad \text{for FP and} \quad x_1 \ge D/2 \quad \text{for ICC and sphere}$$

$$0 \le t \le \hbar_2 \int_0^{x_2} \mathrm{d}x_2/\bar{v}_e \quad \text{and} \quad T^* = \frac{T - T_\infty}{T_S - T_\infty}$$

The solutions of Eq. (6) from Carslaw and Jaeger[15] for the FP, ICC, and sphere, respectively, are as follows:

Flat plat solution

$$T^* = erfc(X/2\sqrt{\alpha t})$$

or

$$T^*|_{t=Y/\bar{v}_e} = erfc(X/2\sqrt{\alpha Y/\bar{v}_e}) \qquad Y > 0 \tag{7}$$

Infinite circular cylinder solution

$$T^* = \sqrt{D/(2r)}\, erfc\left(\frac{r - D/2}{2\sqrt{\alpha t}}\right) + \frac{(r - D/2)\sqrt{\alpha t}}{4r^{3/2}\sqrt{D/2}}\, ierfc\left(\frac{r - D/2}{2\sqrt{\alpha t}}\right) + \frac{(9D^2/4 - Dr - 7r^2)\alpha t}{32r^{5/2}(D/2)^{3/2}}\, i^2erfc\left(\frac{r - D/2}{2\sqrt{\alpha t}}\right)$$

This solution is valid for a short time. However, we are interested in the very short time solution, i.e., $t \to 0$, $Fo_r \ll 1$ $(Pe_r \gg 1)$. Therefore, the first term of the above solution will be taken for this range, i.e.,

$$T^*|_{t=(\theta D)/(2\bar{v}_e)} = \sqrt{D/(2r)}\, erfc\left[\frac{r - D/2}{2\sqrt{(\alpha\theta D)/(2\bar{v}_e)}}\right] \qquad \theta > 0 \tag{8}$$

Sphere solution

$$T^* = \frac{D}{2}\frac{1}{r}\, erfc\left(\frac{r - D/2}{2\sqrt{\alpha t}}\right)$$

or

$$T^*|_{t=(\theta D)/(2\bar{v}_e)} = \frac{D}{2}\frac{1}{r}\, erfc\left[\frac{r - D/2}{2\sqrt{(\alpha\theta D)/(2\bar{v}_e)}}\right] \qquad \theta > 0 \tag{9}$$

The local wall heat flux that is related to the temperature gradient is

$$q_S(x_2) = -k(T_S - T_\infty)\frac{\partial T^*}{h_1 \partial x_1}\bigg|_{\text{surface}} \tag{10}$$

Taking the derivative of Eqs. (7–9) and substituting in Eq. (10) gives the local wall heat flux for three geometries:

$$q_S(Y) = k(T_S - T_\infty)\frac{1}{\sqrt{\pi}\sqrt{\alpha Y/\bar{v}_e}} \qquad \text{for FP} \tag{11}$$

$$q_S(\theta) = \frac{1}{\sqrt{\pi}}\frac{k(T_S - T_\infty)}{\sqrt{(\alpha D\theta)/(2\bar{v}_e)}} \qquad \text{for ICC} \tag{12}$$

$$q_S(\theta) = \frac{k(T_S - T_\infty)}{D/2} + \frac{1}{\sqrt{\pi}}\frac{k(T_S - T_\infty)}{\sqrt{(\alpha D\theta)/(2\bar{v}_e)}} \qquad \text{for sphere} \tag{13}$$

The transient conduction solution provides an analytic solution for the local Nusselt number, $\mathscr{L}q_S/k(T_S - T_\infty)$, which consists of the local boundary-layer term and the linear sum of this term and the constant term corresponding to the diffusive limit $(Re_\mathscr{L} \to 0)$ for the sphere only. Therefore, the area-averaged Nusselt number

$$Nu_\mathscr{L} = \frac{1}{A}\int\int_A \frac{\mathscr{L}q_S(x_2)}{k(T_S - T_\infty)}\, dA$$

is given by

$$Nu_\mathscr{L} = \frac{2}{\sqrt{\pi}}\sqrt{\frac{L\bar{v}_e}{\alpha}} \qquad \text{for FP}$$

$$Nu_D = \sqrt{\frac{2}{\pi}}\frac{2}{\sqrt{\pi}}\sqrt{\frac{D\bar{v}_e}{\alpha}} \qquad \text{for ICC}$$

$$Nu_D = 2 + \frac{1}{\sqrt{2.5}}\frac{2}{\sqrt{\pi}}\sqrt{\frac{D\bar{v}_e}{\alpha}} \qquad \text{for sphere} \tag{14}$$

The area-averaged effective velocity will be determined in the following sections.

## Definition of $\bar{v}_e^\infty$ as $Pr \to \infty$

Table 4 shows step-by-step how one can determine $\bar{v}_e^\infty$ for the different body shapes when $Pr \to \infty$, $Re_\mathscr{L} \gg 1$ and $\delta \gg \delta_T$.

## Definition of $\bar{v}_e^0$ as $Pr \to 0$

Let us consider that the body is immersed in an inviscid flow, i.e., $Pr \to 0$ and $Re_\mathscr{L} \gg 1$. Therefore, the thermal boundary layer (TBL), $\delta_T$, around the body is very thick relative to the hydrodynamic boundary layer (HBL) 6. Therefore, at the edge of the TBL, we have the following relationships at the edge of the HBL:

$$v|_{(\delta + D/2)} = V_\infty \qquad \text{for FP}$$

$$v_\theta|_{(\delta + D/2)} = V_\infty \cdot \{1 + \frac{\quad}{\delta + D/2)} \qquad \sin\theta \qquad \text{for ICC}$$

$$v_\theta|_{(\delta + D/2)} = \frac{V_\infty}{2}\cdot\left\{2 + \left[\frac{\quad}{+ D/2)}\right.\right. \qquad \sin\theta \qquad \text{for sphere} \tag{15}$$

In addition to the usual postulates, the HBL will be presumed to be thin relative to the radius of the cylinder or the sphere. Therefore, $v_\theta|_{\delta + D/2} = V = 2V_\infty \sin\theta$ (for ICC) and $1.5V_\infty \sin\theta$ (for sphere). Furthermore, the local velocity at arbitrary $\theta$ will be considered uniform across the TBL. The area-mean effective velocity is

$$\frac{\bar{v}_e^0}{V_\infty} = \frac{1}{A}\int\int_A \frac{V}{V_\infty}\, dA = C_0 \tag{16}$$

where $C_0 = 1$, 1.273, and 1.178 for the FP, ICC, and sphere, respectively.

## Definition of $\bar{v}_e$ for all $Pr$

The method of Churchill and Usagi[17] "blending technique" will be used to develop a general expression for $\bar{v}_e$ valid for all $Pr$. The effective velocity can be determined in the following form based on the study of Refai Ahmed and Yovanovich[11]:

$$\bar{v}_e = \frac{\bar{v}_e^\infty}{[1 + (\bar{v}_e^\infty/\bar{v}_e^0)^n]^{1/n}} \tag{17}$$

Substituting $\bar{v}_e^0$ and $\bar{v}_e^\infty$ in Eq. (17) gives the effective velocity as a function of Prandtl number, power-law parameter $\gamma$, and blending parameter $n$

$$\frac{v_e}{V_\infty} = \frac{C_0/[(2\gamma + 1)Pr^{1/3}]}{\{1 + [1/(2\gamma + 1)Pr^{1/3}]^n\}^{1/n}} \tag{18}$$

where $0 < \gamma < 1$ and $0 < Pr < \infty$. The values of $n$ and $\gamma$ will be determined in the following section.

**Table 4 Definitions of $\bar{v}_c^\infty$ as $Pr \to \infty$ and $Re_L \gg 1$**

| Step | FP | ICC | Sphere |
|---|---|---|---|
| 1. Continuity equation inside the HBL or TBL thicknesses | | $\nabla \cdot \boldsymbol{u} = 0$ | |
| 2. Apply scaling analysis inside the HBL thickness | $u\|_\delta \sim V_\infty(\delta/L)$ | $v_r\|_{\delta+D/2} \sim 2(\delta/D)\cdot[V(\theta)/\theta]$ | |
| 3. Applying scaling analysis inside the TBL thickness | $u\|_{\delta_T} \sim V_\infty(\delta_T^2/L\delta)$ | $v_r\|_{\delta_T+D/2} \sim 2(\delta_T^2/D\delta)\cdot[V(\theta)/\theta]$ | |
| 4. Momentum equation inside the HBL thickness | | $\nabla \cdot (u_i u_2) = \delta_{i2}\nabla P + \nu\nabla^2 u_2$ | |
| 5. Apply scaling analysis inside the TBL thickness | $(\delta/L) \sim \sqrt{1/Re_L}$ | $(\delta/D) \sim \sqrt{\theta/2Re_D(\theta)}$ | |
| 6. Energy equation inside the TBL thickness | | $\nabla \cdot (u_i T) = \alpha\nabla^2 \cdot T$ | |
| 7. Apply scaling analysis inside the TBL thickness | $(\delta/L) \sim (1/Pr^{1/3}\sqrt{Re_L})$ | $(\delta/D) \sim (1/Pr^{1/3})\sqrt{\theta/2Re_D(\theta)}$ | |
| 8. Ratio of TBL to HBL thicknesses, $\delta_T/\delta$ | $(V/V_\infty) \sim (\delta_T/\delta) \sim (1/Pr^{1/3})$ | $[V/V(\theta)] \sim (\delta_T/\delta) \sim (1/Pr^{1/3})$ | |
| 9. Momentum flux inside TBL thickness | | $\frac{\rho}{\delta_T}\int_0^{\delta_T} u_2(V - v_2)\,dx_1$ | |
| 10. Assume the flow has a uniform local effective velocity (function of $\theta$ for the ICC and sphere) | | $\frac{\rho}{\delta_T}\int_0^{\delta_T} v_c^\infty(x_2)\cdot(V - v_2)\,dx_1$ | |
| 11. Equate steps 9 and 10, and rewrite it as a function of the momentum and displacement thicknesses | | $\frac{v_c^\infty(x_2)}{V} = \frac{\delta_M^T}{\delta_D^T}$ | |
| 12. The last step can be expressed in the following form using scaling analysis | $v_c^\infty(x_2) = (V_\infty/Pr^{1/3})(\delta_M^T/\delta_D^T)$ | $v_c^\infty(x_2) = (V(\theta)/Pr^{1/3})(\delta_M^T/\delta_D^T)$ | |
| 13. The velocity profile can be expressed in the power-law form | $(v/V) = (x_1/\delta_T)^\gamma$ | $(v_\theta/V) = \{[x_1 - D/2)]/\delta_T\}^\gamma$ | |
| 14. Use similarity parameter, $\eta = y^a/\delta_T$ and integrate the velocity over the surface area. In addition the equations of $V(\theta)$ are obtained from the ideal flow solution, therefore. $\bar{v}_c^\infty$ arc | $\frac{V_\infty}{(2\gamma + 1)Pr^{1/3}}$ | $\frac{1.273V_\infty}{(2\gamma + 1)Pr^{1/3}}$ | $\frac{1.178V_\infty}{(2\gamma + 1)Pr^{1/3}}$ |

$^a y = x_1 - D/2$ for ICC and sphere. $y = x_1$ for FP

## Results and Discussion

In order to determine the equation of $Nu_L$, one can substitute Eq. (18) into Eqs. (14), respectively. The area-averaged Nusselt number can be expressed in the following forms:

$$Nu_L = 1.128Re_L^{1/2}\frac{Pr^{1/3}/\sqrt{2\gamma + 1}}{\left\{1 + \left[\frac{1.0}{(2\gamma + 1)^3 Pr}\right]^{n/3}\right\}^{1/(2n)}} \quad \text{for FP}$$

$$Nu_D = 1.015Re_D^{1/2}\frac{Pr^{1/3}/\sqrt{2\gamma + 1}}{\left\{1 + \left[\frac{1.0}{(2\gamma + 1)^3 Pr}\right]^{n/3}\right\}^{1/(2n)}} \quad \text{for ICC}$$

$$Nu_D = 2 + 0.775Re_D^{1/2}\frac{Pr^{1/3}/\sqrt{2\gamma + 1}}{\left\{1 + \left[\frac{1.0}{(2\gamma + 1)^3 Pr}\right]^{n/3}\right\}^{1/(2n)}}$$

$$\text{for sphere} \qquad (19)$$

It has been found that $n = 3$ gives the best fit by matching Eqs. (19) with available air data.” In addition, it was found that simple correlations for $\gamma$ as $F(Re_L)$ based on the previous works are given by

$$\gamma_L = \frac{1}{[1 + Re_L^{1.25}]^{1/5}} \quad \text{for FP}$$

$$\gamma_D = \frac{1}{[1 + (Re_D^{0.75}/300)^5]^{1/5}} \quad \text{for ICC}$$

$$\gamma_D = \frac{1}{[1 + Re_D^{1.25}]^{1/5}} \quad \text{for sphere} \qquad (20)$$

## Flat Plate Model and Data Comparison

Figure 2 shows the relationship between $\{(Nu_L - Nu_L^0)/[F(Pr, \gamma_L)\sqrt{Re_L}]\}$ ($Nu^0$: for both the FP and ICC is 0) and $Re_L$. In addition, Fig. 2 shows the comparisons between the present model and the previous studies[6,8,9,14,17–29] for various Prandtl numbers.

Figure 2a shows the comparisons between the present approximate analytical solution [Eq. (19)], for the FP, and the previous analytical and experimental investigations. The analytical studies such as Pohlhausen[17] and Levy[18] are in a very good agreement with the present model at low Reynolds numbers. However, at high $Re_L = 10^4$, the maximum difference is within 30% ($Pr = 0.7$) and 35% ($Pr = 10$). On the other hand, the maximum difference between the present model and the Pohlhausen solution for $Pr = 0.01$ is found within 7%. In addition, it is found that the maximum difference between the present model [Eq. (19)], and the solutions of both Pohlhausen and Levy is 4%, if $\gamma = 1$, i.e., the velocity profile is linear across the thermal boundary layer.

Figure 2a also shows the comparisons between the experimental studies of Slegel and Hawkins[4] and Parmalee and Huebscher,[5] and the analytical solution” for high $Re_L$ and the present model. It is interesting to observe from these comparisons that the Parmalee and Huebscher[5] data are below the Slegel and Hawkins' data by 40–50%. Furthermore, Knudsen and Katz[6] obtained agreement only with Slegel and Hawkins.[4]

## Circular Cylinder Model and Data Comparison

Figure 2b shows the comparisons between the present model [Eq. (19)], for the ICC and the previous studies. One observes that the previous studies of Churchill and Bernstein,” Morgan,” King,[19] Kramers,[20] and Krall and Eckert” correlations are in a very good agreement with the present model, 5–7%,

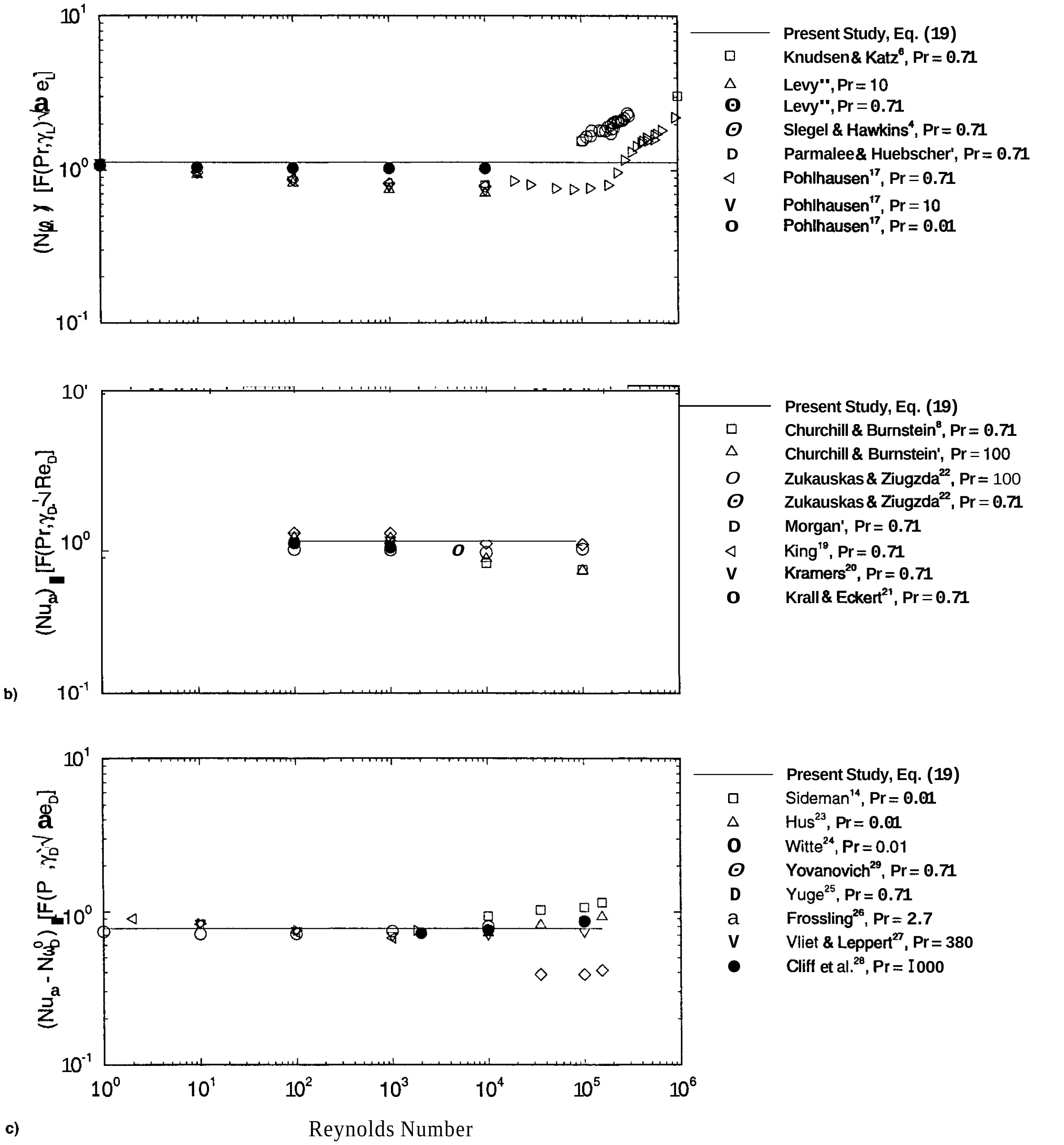


**Fig. 2 Comparison between present approximate analytical model and previous studies for different Prandtl numbers: a) flat plate, b) infinite circular cylinder, and c) sphere.**

in the range of $Re_D$ from $10^2$ to $10^3$. In addition, the correlation of Zukauskas and Ziugzda'' shows the same agreement with the present model at $Pr = 10^2$, but for $Pr = 0.71$ there is a difference of 12%. Furthermore, the study of Zukauskas and Ziugzda'' is in very good agreement up to $Re_D = 10'$ (maximum difference is 5%). The Churchill and Bernstein[8] correlation has a maximum difference of 35% with the present model at $Re_D = 10'$. Achenbach''' introduced experimental relationships between $Nu_D$ and $Re_D$ over the range $3 \times 10^4 < Re_D < 4 \times 10^6$, for $L/D = 1.36$. Achenbach''' indicated that his experimental data were higher than Hilpert,''' and this is due to blockage and low span ratio $L/D$ effects in Hilpert.[10] Unfortunately, the present model cannot be compared against Achenbach,''' where this model is valid up to $Re_D = 10'$.

## Sphere Model and Data Comparison

Figure 2c shows the comparisons between the present model [Eq. (19)], for the sphere and previous studies[4–6,8,9,14,17–29] for various Prandtl numbers. One observes that the maximum difference between the previous studies correlations and the present model in the range of $1 < Re_D < 10'$ is approximately 11%, which generally occurs at $Re_D = 10'$. On the other hand, there is almost a 33% difference between the present model and that of Sideman.'? The main reason for this is that Sideman[14] approximated the convective term of the energy equation by assuming $(V_\infty/r)(\partial T/\partial \theta)$, and neglecting the curvature of the sphere with respect to the radial conduction. Achenbach'' obtained experimental data up to $Re_D = 5 \times 10^6$ and he had good agreement with Yuge.[25] Therefore, the present investigation is also in agreement with Achenbach'' (where the present study has good agreement with Yuge[25] as shown in Fig. 2c).

Finally, one can conclude from Fig. 2 that the constant C, for the flat plate, infinite circular cylinder, and sphere are 1.128, 1.015. and 0.775, respectively. In addition, $F(Pr, \gamma_f)$ has the general form

$$\frac{Pr^{1/3}}{[(2\gamma_f + 1)^3 + 1/Pr]^{1/6}}$$

for the FP. ICC, and sphere.

## Design Correlations

The approximate analytical solutions are developed based on the scale length of L for FP and $D$ for ICC. However, it is important to introduce a general solution for plates and circular cylinders. Therefore, Eq. (19) will be rewritten based on $\sqrt{A}$ as follows:

$$Nu_{\sqrt{A}} = 1.128(W/L)^{0.25}Re_{\sqrt{A}}^{0.5}F(Pr, \gamma_{\sqrt{A}}) \qquad \text{FP} \qquad (21a)$$

$$= 1.015(L/D)^{0.25}Re_{\sqrt{A}}^{0.5}F(Pr, \gamma_{\sqrt{A}}) \qquad \text{ICC} \qquad (21b)$$

$$= 3.545 + 1.032Re_{\sqrt{A}}^{0.5}F(Pr, \gamma_{\sqrt{A}}) \qquad \text{sphere} \qquad (21c)$$

The ratio of $(\sqrt{A}/D)$ is of order $\sqrt{(L/D)}$ as $L/D \to \infty$.

The equations of FP and ICC become functions of $W/L$ and $L/D$, respectively. Therefore, the diffusive limit must be included in the solutions as follows:

$$Nu_{\sqrt{A}} = Nu_{\sqrt{A}}^0 + 1.128(W/L)^{0.25}Re_{\sqrt{A}}^{0.5}F(Pr, \gamma_{\sqrt{A}})$$

$$\text{rectangular plate} \qquad (22a)$$

$$= Nu_{\sqrt{A}}^0 + 1.015(L/D)^{0.25}Re_{\sqrt{A}}^{0.5}F(Pr, \gamma_{\sqrt{A}})$$

$$\text{circular cylinder} \qquad (22b)$$

The diffusive limits for circular cylinders and rectangular plates can be estimated, within ±5%, as follows (more details in Ahmed[32] and Yovanovich[11]):

Circular cylinders

$$Nu_{\sqrt{A}}^0 = 3.5(L/D)^{0.02} \qquad 0 < L/D \le 1$$

$$= 3.385 + 0.082(L/D) \qquad 1 < L/D \le 8 \qquad (23)$$

$$= 4\sqrt{L/D}/\ln 2(L/D) \qquad 8 < L/D \qquad (24)$$

Rectangular plates

$$Nu_{\sqrt{A}}^0 = \frac{8.89\sqrt{W/L}}{\ln 4W/L} \qquad 0.0 \le L/W \le 0.25$$

$$Nu_{\sqrt{A}}^0 = \frac{(1.0 + \sqrt{L/W})^2}{\sqrt{0.5L/W}} \qquad 0.25 \le L/W \le 1 \qquad (25)$$

## Summary and Conclusions

A forced convection heat transfer method based on an approximate analytical approach is developed for simple body shapes such as the flat plate, infinite circular cylinder, and sphere. This method is valid for the range of Reynolds number, $0 < Re_L < 10^5$, and all values of the Prandtl number. In addition, the present method considers the effect of the velocity profile as a function of Reynolds number on the forced convection heat transfer. Also, this technique is shown to be in very good agreement with numerous previous investigations. Furthermore, $F(Pr, \gamma_f)$ was obtained in a general form that is valid for the FP, ICC, and sphere. Finally, this investigation has led to design correlations for circular cylinders and rectangular plates [Eq. (22)], which can be applied in a wide range of Reynolds number and all Prandtl numbers.

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## References

[1]Yovanovich, M. M., Lee, S., and Gayowsky, T. J., "Approximate Analytical Solution of Laminar Forced Convection Heat from an Isothermal Plate," AIAA Paper 92-0248, 1992.

[2]Kays, W. M., and Crawford, M. E., *Convection Heat and Mass Transfer,* 2nd ed., McGraw–Hill, New York, 1980.

[3]Schlichting, H., *Boundary Layer Theory,* 6th ed., McGraw–Hill, New York, 1968.

[4]Slegel, L., and Hawkins, G., 1946; also see Ref. 6.

[5]Parmalee, G. V., and Huebscher, R. G., 1947; also see Ref. 6.

[6]Knudsen, J., and Katz, D., *Fluid Dynamics and Heat Transfer,* McGraw–Hill, New York, 1958.

[7]Ahmad, R. A., and Qureshi, Z. H., "Laminar Mixed Convection from a Uniform Heat Flux Cylinder," AIAA Paper 91-0383, 1991.

[8]Churchill, S. W., and Bernstein, M., "A Correlating Equation for Forced Convection from Gases and Liquids to a Circular Cylinder in Crossflow," *Journal of Heat Transfer,* Vol. 99, No. 2, 1977, pp. 300–306.

[9]Morgan, V. T., "The Overall Convection Heat Transfer from Smooth Circular Cylinders," *Advances in Heat Transfer,* Vol. 11, Academic, New York, 1975, pp. 199–264.

[10]Hilpert, R., 1933; also see Ref. 9.

[11]Ahmed, G. R., and Yovanovich, M. M., "Approximate Analytical Solution of Forced Convection Heat Transfer from Isothermal Spheres for all Prandtl Numbers," *Journal of Heat Transfer,* Vol. 116, No. 4, 1994, pp. 838–843.

[12]Happel, J., and Brenner, H., *Low Reynolds Number Hydrodynamics,* 2nd ed., Noordhoff International, Leyden, The Netherlands, 1973.

[13]Jafarpur, K., "Analytical and Experimental Study of Laminar Free Convection Heat Transfer from Isothermal Convex Bodies of Arbitrary Shapes," Ph.D. Dissertation, Dept. of Mechanical Engineering, Univ. of Waterloo, Waterloo, Ontario, Canada, 1992.

[14]Sideman, S., "The Equivalence of the Penetration Theory and Potential Flow Theories," *Industrial and Engineering Chemistry,* Vol. 58, No. 2, 1966, pp. 54–58.

[15]Carslaw, H. S., and Jaeger, J. C., *Conduction of Heat in Solids,* 2nd ed., Clarendon, Oxford, England, UK, 1959.

[16]Churchill, S. W., and Usagi, R., "A General Expression for the Correlation of Rates of Transfer and Other Phenomena," *AIChE Journal,* Vol. 18, 1972, pp. 1121–1132.

[17]Pohlhausen, E., 1921; see also Ref. 2.

[18]Levy, S., 1952; see also Ref 6.

[19]King, L., 1959; see also Ref. 9.

[20]Kramers, H., "Heat Transfer from Spheres to Flowing Media," *Physica,* Vol. 12, Nos. 2 and 3, 1946, pp. 61–81.

[21]Krall, K. M., and Eckert, E. R., "Local Heat Transfer Around a Cylinder at Low Reynolds Number," *Journal of Heat Transfer,* Vol. 95, No. 2, 1973, pp. 273–275.

[22]Zukauskas, A., and Ziugzda, J., *Heat Transfer of a Cylinder in Crossflow,* translated by E. Bagdonaite, edited by G. F. Hewitt, Hemisphere, Washington, DC, 1985.

[23]Hsu, S. T., *Engineering Heat Transfer,* D. Van Nostrand, New York, 1963.

[24]Witte, L. C., "An Experimental Study of Forced Convection Heat Transfer from a Sphere to Liquid Sodium," *Journal of Heat Transfer,* Vol. 90, 1968, pp. 9–12.

[25]Yuge, T., "Experiments on Heat Transfer from Spheres Including Combined Natural Convection," *Journal of Hear Transfer,* Vol. 82, 1960, pp. 214–220.

[26]Frössling, N. M., "The Evaporation of Falling Drops," *Gerlands Beitr. Geophys.,* Vol. 52, 1938, pp. 170–216.

[27]Vliet, G. C., and Leppert, G., "Forced Convection Heat Transfer from an Isothermal Sphere to Water," *Journal of Heat Transfer,* Vol. 83. 1961, pp. 163–175.

[28]Clift, R., Grace, J. R., and Weber, M. E., *Bubbles, Drops and Particles,* Academic, New York, 1978.

[29]Yovanovich, M. M., "General Expression for Forced Convection Heat and Mass Transfer from Isopotential Spheroids," AIAA Paper

88-0743, 1988.

[30]Achenbach, E., "Total and Local Heat Transfer from a Smooth Circular Cylinder in Cross-Flow at High Reynolds Number," *International Journal of Heat and Mass Transfer*, Vol. 18, 1975, pp. 1387–1395.

[31]Achenbach, E., "Heat Transfer from Spheres up to $Re = 6 \times 10^6$," *Sixth International Heat Transfer Conference*, Vol. 5, Toronto, Ontario, Canada, 1978, pp. 341–346.

[32]Refai Ahmed, G., "Experimental and Approximate Analytical Study of Forced Convection Heat Transfer from Isothermal Body Shapes," Ph.D. Dissertation, Dept. of Mechanical Engineering, Univ. of Waterloo, Waterloo, Ontario, Canada 1994.

[33]Yovanovich, M. M., "Dimensionless Shape Factors and Diffusion Lengths of Three-Dimensional Bodies," *Proceedings of the ASME/JSME Thermal Engineering Joint Conference*, Vol. 1, 1995, pp. 103–113.