MODELS OF LAMINAR NATURAL CONVECTION FROM VERTICAL AND HORIZONTAL ISOTHERMAL CUBOIDS FOR ALL PRANDTL NUMBERS AND ALL RAYLEIGH NUMBERS BELOW 10^{11}

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ABSTRACT

A simple general model is developed to predict natural convection heat transfer from isothermal horizontal and vertical cuboids of arbitrary shape for a wide range of Rayleigh and Prandtl numbers. The model is based on the linear superposition of the diffusive limit valid for a stagnant medium and the laminar bounday layer limit. The laminar boundary layer limit consists of two parameters: i) the Prandtl number function which is valid for all values of the Prandtl number, and ii) the body-gravity function which accounts for the buoyancy-induced fluid flow over the entire surface of the body. Methods are presented for estimating the diffusive limit and the body-gravity functions for a general cuboid. Expressions are developed for several special cases of cuboids. The general cuboid model predictions are compared against air data for several cuboids for a wide range of Rayleigh numbers. There is very good agreement between the present cuboid model and air data. The RMS and maximum percent differences are found to be in the ranges: $2.5 - 6.1$ and $3.8 - 12.1$ respectively.

NOMENCLATURE

A	=	surface area of the body; m^2
$\tilde{A}$	=	area fraction
$\tilde{A}_i$	=	area fraction of the ith component
$\sqrt{A}$	=	characteristic length of the body proposed by Yovanovich (1987a); m
AR	=	aspect ratio of body
a, b, c	=	semiaxes of ellipsoid; $a \geq b \geq c$
C	=	capacitance: charge per unit potential
$C^{\star}_{\sqrt{A}}$	=	dimensionless capacitance
c_p	=	specific heat at constant pressure; $J/kg{\cdot}K$
D	=	cylinder diameter, m
$E(\phi, \kappa)$	=	incomplete elliptic integral of second kind
e	=	eccentricity
$F(Pr)$	=	Prandtl number function (Churchill and Churchill,1975), $\left[0.670/[1 + (0.50/Pr)^{9/16}]^{4/9}\right]$
$F(\phi, \kappa)$	=	incomplete elliptic integral of the first kind
g	=	scalar gravitational acceleration; m/s^2
$Gr_{\sqrt{A}}$	=	Grashof number, $g\beta(T_s - T_\infty)(\sqrt{A})^3/\nu^2$
h	=	heat transfer coefficient; $W/m^2{\cdot}K$
H	=	cuboid height; m
K	=	complete elliptic integral of the first kind
k	=	thermal conductivity; $W/m{\cdot}K$
L	=	cuboid length; m
$\vec{n}$	=	outward surface normal vector
$Nu_{\sqrt{A}}$	=	Nusselt number, $h\sqrt{A}/k$
$Nu^{\infty}_{\sqrt{A}}$	=	diffusive limit; $Nu^{\infty}_{\sqrt{A}} = S^{\star}_{\sqrt{A}}$
$P(\theta)$	=	local perimeter; m
P, P_{max}	=	perimeter of projected area onto a horizontal plane, maximum perimeter; m

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Pr	=	Prandtl number; ν/α
Q	=	total heat flow rate; W
$Q^\star_{\sqrt{A}}$	=	dimensionless heat flow rate
R	=	thermal resistance; K/W
$R^\star_{\sqrt{A}}$	=	dimensionless thermal resistance
$Ra_{\sqrt{A}}$	=	Rayleigh number, $Gr_{\sqrt{A}}Pr$
Ra_w	=	Rayleigh number, $Gr_w Pr$
RMS	=	Root-Mean-Square value
$S^\star_{\sqrt{A}}$	=	dimenionless shape factor
Sc	=	Schmidt number
Sh_w	=	Sherwood number based on w
Sh_0	=	Sherwood number based on w at $Ra_w = 0$
T	=	temperature; K
T_f	=	film temperature, $(T_s + T_\infty)/2$; K
u, v	=	aspect ratios of ellipsoid; $u = c/a, v = b/a$
W	=	cuboid width; m

Subscripts

$\sqrt{A}$	=	based on $\sqrt{A}$, as the characteristic length
s	=	at the surface
∞	=	at a remote point from the body
max, mid, min	=	maximum, middle and minimum

Superscripts

$\rightarrow$	=	vector quantity
∞	=	estimated at $Ra \rightarrow 0$
$\tilde{}$	=	dimensionless quantity
$*$	=	dimensionless quantity

GreekSymbols

α	=	thermal diffusivity, $k/\rho c_p$; m
β	=	volumetric expansion coefficient; K^{-1}
κ	=	modulus of incomplete elliptic integral
ϕ	=	amplitude of incomplete elliptic integral
θ	=	angle between gravity vector and outward normal to surface; rad
ν	=	kinematic viscosity, μ/ρ; m^2/s
ρ	=	density; kg/m^3

Miscellaneous

bot	=	abbreviation for bottom surface
cuboid	=	value for entire cuboid
cube	=	value for cube
long prism	=	value for long prism
rect plate	=	value for rectangular plate
sq plate	=	value for square plate
top	=	abbreviation for top surface
vert plate	=	value for vertical plates
vert sq prism	=	value for vertical square prism

INTRODUCTION

For more than nine decades laminar natural convection heat (and mass) transfer from isothermal horizontal cubes, horizontal and vertical square and rectangular prisms; finite plates of square or rectangular shape (planform) in various orientations (inclinations from vertical to horizontal with respect to the gravity vector) has been the subject of numerous experimental, theoretical and numerical studies.

Each study was limited to one or perhaps two aspects of the total possible cases which frequently occur in today's technology. Examples found in the microelectronic industry are rectangular heat sources of different aspect ratios which are oriented horizontally or vertically, and they are cooled by natural convection heat transfer through all faces. Depending on the aspect ratios of the heat sources, the heat transfer rates can differ by as much as 60 % for the same total heat transfer area.

There is at present no simple, single model which can predict accurately Nusselt numbers for i) all aspect ratios, ii) for all Prandtl numbers, and iii) all Rayleigh numbers below 10^{11}, (laminar flow range).

The main objective of this work is to develop a simple three-dimensional cuboid (rectangular parallelopiped) model to predict laminar natural convection heat transfer from horizontal or vertical isothermal cubes, square and rectangular prisms which reduce to square and rectangular plates as limiting cases (which are very important limits in many applications).

The proposed model will be based on the linear superposition of the diffusive limit and the laminar boundary layer limit. The diffusive limit is directly related to the dimensionless shape factor and several novel methods of estimating the shape factor of cuboids will be presented. The boundary layer limit consists of the product of the Prandtl number function, the body-gravity function and the Rayleigh number to a fixed value of the exponent. General and specific expressions for the body-gravity function will be presented for the cuboid and several related special cases.

The cuboid model(s) will be applicable for all fluids ($0 < Pr < \infty$) over a wide range of Rayleigh number ($0 \le Ra < 10^{11}$).

The second objective is to demonstrate the accuracy of the proposed model(s) for natural convection cooling into air from several cuboids over a wide range of Rayleigh number.

REVIEW OF CUBOID MODELS

There is only one paper which deals with a general solution for predicting mass and heat transfer from a horizontal cuboid (Worthington et al., 1987). Another paper (Hassani and Hollands, 1989) presents a general model for predicting natural convection from three-dimensional isothermal bodies of arbitrary shape, and therefore it can be used to model cuboids.

Worthington et al. (1987) developed two semiempirical correlation equations for mass transfer from horizontal cuboids with side dimensions h, w, l; and with the gravity vector parallel to the h side and perpendicular to the $(w\ l)$ surface. They further introduced two geometric parameters: $V = h/w$ and $H = l/w$ which are called horizontal and vertical "shape factors." After a review of the characteristic body lengths used by other investigators they chose to define the Sherwood and Rayleigh numbers with respect to the smaller dimension of the surface perpendicular to the gravity vector: Sh_w and Ra_w.

They included the constant Sh_0 in their correlation equation to "represent the purely diffusive Sherwood number in still fluid." They further stated that "It is useful to include the stagnant medium condition in correlations for three-dimensional objects as its contribution to the overall transfer rate can become significant at low Rayleigh numbers and its omission can affect the exponent on the Rayleigh number." The inclusion of Sh_0 to account for the stagnant medium enabled them to obtain a best fit Rayleigh number exponent very close to 0.25.

Values of Sh_0 for fourteen different cuboid shapes: $0.60 \le V \le 2.03$ and $0.99 \le H \le 5.79$ were calculated using the method outlined by Clift et al. (1978). The values were not given in their paper. This method is an approximation which can lead to significant errors (Jafarpur, 1992).

Mass transfer experiments for Rayleigh numbers between 10^5 and 10^{11} for Schmidt numbers in the range $2300 - 3500$ were conducted for fourteen cuboids.

They introduced an empirical parameter f which they called the interference factor to account for the complex fluid flow over the bottom, sides and top surfaces. Values of f were calculated for each cuboid by a least square fit of the proposed correlation equations to the data. They found that f varied between 0.74 for the tallest square plan cuboid ($V = H = 3$) and 0.50 for the least tall and longest cuboid ($V = 1, H = 4$). The average value for the four cubes tested was calculated to be 0.72. They reported that the cube data and the proposed correlation equation compared well over the range: $10^6 < Ra_w < 10^{10}$ with a maximum deviation of 8 %.

They presented contour plots of the interference factor f for $1 \le H \le 6$ and $0.48 < V < 0.75$. They stated that the use of the contour plots together with the two proposed correlation equations should predict natural convection mass transfer at cuboids in the range of the geometries investigated to within 10 %.

The proposed correlation equations cannot be used for thin horizontal cuboids $0 < V < 0.48$; very tall cuboids $V > 0.75$, and very long horizontal cuboids $6 < H$. The correlation equations are therefore limited to a narrow range of cuboid shapes.

Based on their mass transfer work Worthington et al. (1987) proposed the following two correlation equations for natural convection heat transfer from horizontal isothermal cuboids:

$$Nu_w - Nu_0 = \left[\frac{f(0.675(1+H)V^{3/4} + 0.321H) + 0.161H}{H + V + VH}\right] \cdot (Ra_w \mathcal{F}(Pr))^{0.25} \qquad (1)$$

and

$$Nu_w - Nu_0 = \left[\frac{0.675f(1+H)V^{3/4} + (0.161 + 0.075(Ra_w \mathcal{F}(Pr))^{1/12})H}{H + V + VH}\right] \cdot (Ra_w \mathcal{F}(Pr))^{0.25} \qquad (2)$$

The first equation is restricted to laminar flow and the second is reported to be valid for turbulent flow through the inclusion of $Ra_w^{1/12}$ in Eq. (2).

The Prandtl number function in Eqs. (1) and (2) is defined as

$$\mathcal{F}(Pr) = \left[1 + \left(\frac{0.5}{Pr}\right)^{4/16)}\right]^{-16/9} \qquad (3)$$

which is based on the recommendation of Churchill and Churchill, (1975). The proposed correlation equations were tested against the heat transfer data of a cube cooled by air (Stretton, 1985) in the low range of Rayleigh number: 10^3 to 10^7; and their own heat transfer experiments on air that covered the "transition region," $10^6 \le Ra_w \le 3 \times 10^7$. Good agreement between the model and data was reported for $10^3 < Ra_w < 10^{11}$.

This cuboid model is limited to a narrow range of the geometric parameters: (V, H). It requires information on the empirical parameter f which is available only in graphical form for a narrow range of V and H. This cuboid model cannot predict heat transfer from horizontal and vertical rectangular plates $H \to 0$ which are essential for the modelling of the base-plates and fins of heat sinks. This cuboid model is based on the smaller dimension of the horizontal rectangular surface. This is an arbitrary choice which, therefore, limits the usefulness of the model and it will fail when $w \to 0$ as the cuboid approaches a vertical rectangular plate.

The model of Hassani and Hollands (1989) is based on the blending of three components: i) the diffusive limit, ii) the laminar boundary layer asymptote, and iii) a term which is asserted to account for turbulence from the topmost parts of the complex bodies. The diffusive limit is the one recommended by Yovanovich (1987a).

The proposed method is based on two characteristic lengths; there are two blending parameters which are semiempirical; and several steps are required to execute the method.

The method has been verified by comparison of the model predictions against air data obtained from several complex body shapes (Chamberlain, 1983; and Hassani, 1987) some of which are cuboids.

The agreement between the model predictions and the data over several decades of the Rayleigh number are reported to be very good with RMS and maximum percent differences less than 6 % and 13 % respectively. The largest differences were observed with thin oblate spheroids; thin horizontal circular and square disks; and the horizontal cube. The aspect ratio of the oblate shperoid and the circular and square disks was 0.1.

The Hassani-Hollands method will fail when the vertical dimension is zero; therefore the method is not recommended for horizontal rectangular plates.

The Worthington et al. (1987) model and the Hassani and Hollands method (1989) are not simple; they do not handle horizontal rectangular plates; and they are somewhat difficult to use and to implement into heatsink models.

GENERAL THREE-DIMENSIONAL NATURAL CONVECTION MODEL

The general expression for natural convection heat transfer from isothermal three-dimensional bodies was proposed by Yovanovich (1987a, 1987b, 1988) in the form:

$$Nu_{\sqrt{A}} = Nu^{\infty}_{\sqrt{A}} + F(Pr)\, G_{\sqrt{A}}\, Ra^{1/4}_{\sqrt{A}} \tag{4}$$

which is the linear superposition of the diffusive limit $Nu^{\infty}_{\sqrt{A}}$ corresponding to $Ra_{\sqrt{A}} = 0$ and the laminar boundary-layer asymptote $F(Pr)\, G_{\sqrt{A}}\, Ra^{1/4}_{\sqrt{A}}$ which is valid in the range $10^4 < Ra_{\sqrt{A}} < 10^{11}$. The laminar boundary-layer asymptote consists of the product of the Prandtl number function $F(Pr)$, the body-gravity function $G_{\sqrt{A}}$, and the Rayleigh number $Ra_{\sqrt{A}}$. The characteristic length is the square-root of the total active surface area $\sqrt{A}$ which was first proposed by Yovanovich (1987a, 198b, 1988) for natural and forced convection heat transfer from bodies of arbitrary shape. This characteristic length has been used by Yovanovich to nondimensionalize thermal constriction results of numerous geometries. Recently Hassani and Hollands (1989a, 1989b) have adopted this characteristic length for the Nusselt number and the Nusselt number corresponding to the diffusive limit in their work on natural convection heat transfer from three-dimensional bodies. They used another complex length scale for the Rayleigh number.

The laminar Prandtl number function of Churchill and Churchill (1975)

$$F(Pr) = \frac{0.670}{\left[1 + (0.5/Pr)^{9/16}\right]^{4/9}} \tag{5}$$

is recommended as the *universal* function valid for all geometries. The body-gravity function of Lee-Yovanovich-Jafarpur (1991)

$$G_{\sqrt{A}} = \left[\frac{1}{A}\iint_A \left(\frac{P(\theta)}{\sqrt{A}}\sin\theta\right)^{1/3} dA\right]^{3/4} \tag{6}$$

is recommended for axisymmetric and two-dimensional geometries.

The proposed three-dimensional model, Eq. (3), has been validated experimentally for a range of body shapes such as i) axisymmetric spheroids (oblate and prolate), sphere, ii) two-dimensional elliptic and circular cylinders, iii) thin circular and square plates in the vertical and horizontal orientation, and iv) other body shapes (cube, cones with apex facing upward and downward).

The body-gravity function, Eq. (5), cannot be used in its present form for bodies which are thin and have horizontal surfaces facing upward or downward because $\sin\theta$ is zero and therefore $G_{\sqrt{A}} = 0$ for these important cases.

Semiempirical methods will be used to obtain body-gravity functions for horizontal surfaces facing upward or downward.

New general expressions will be developed in the following sections for the diffusive limit and the body-gravity function for cuboids which are complex three-dimensional bodies which possess corners, edges and horizontal and vertical surfaces.

BODY-GRAVITY FUNCTION FOR COMPLEX BODY SHAPES

Buoyancy-induced flow over complex body shapes can be modelled by i) decomposing the total body surface into component surfaces corresponding to the fluid flow, and ii) using the general formula, Eq. (6), for each component surface to find the corresponding composite body-gravity function $G_{\sqrt{A}}$.

The overall body-gravity function for the total body surface is then determined by combining the component surfaces A_i and their respective body-gravity functions $G_{\sqrt{A_i}}$ into a composite value. Equation (6) can be used for all surfaces except horizontal surfaces ($\sin\theta = 0$) facing upward or downward. At present semiempirical methods must be used to model buoyancy-induced flow over horizontal surfaces (Jafarpur 1992).

There are two important flow arrangments for which the composite or overall body-gravity function can be determined with relative ease. These are complex bodies such as a circular cylinder with hemispherical ends which is placed in a large extent of air in either the horizontal (axis perpendicular to the gravity vector) or vertical (axis parallel to the gravity vector) orientations.

In the first orientation the two ends and the horizontal surface are cooled by *different flows* of air and the component surfaces are said to be in the *parallel flow arrangement.* In the second orientation the component surfaces are cooled by the same fluid flow which starts at the lower stagnation point, flows over the lower hemispherical end, then over the vertical cylindrical surface, and finally over the top hemispherical end. In this case the component surfaces are said to be in the *series flow arrangement.*

The above method of decomposing a complex body shape into parallel or series flow arrangements can be applied to many interesting natural convection problems. Some orientations such as inclined short cylinders with flat ends or hemispherical ends, or inclined cuboids are more difficult to model.

If the buoyancy-induced flow over a complex body shape can be decomposed into N component surfaces with area-fractions $\tilde{A}_i$, and the corresponding body-gravity functions $G_{\sqrt{A_i}}$ can be determined, then the composite body-gravity function for the entire body surface can be evaluated by means of the following parallel flow arrangement formula (Lee-Yovanovich-Jafarpur, 1991):

$$G_{\sqrt{A}} = \sum_{i=1}^{N} G_{\sqrt{A_i}} \tilde{A}_i^{7/8} \tag{7}$$

On the other-hand if the total body surface can be decomposed according to the series flow arrangement as described above, the composite body-gravity function can be evaluated by means of the following series flow arrangement formula (Lee-Yovanovich-Jafarpur, 1991):

$$G_{\sqrt{A}} = \left[\sum_{i=1}^{N} G_{\sqrt{A_i}}^{4/3} \tilde{A}_i^{7/6}\right]^{3/4} \tag{8}$$

For two-dimensional surfaces, such as vertical disks or plates of arbitrary shape with variable perimeter $P(z)$ the body-gravity function can be easily obtained from the following simple formula which was derived from Eq. (6) after setting $\sin\theta = 1$:

$$G_{\sqrt{A}} = \frac{2}{A^{7/8}} \int_0^{P_{max}/2} [S(z)]^{3/4} \, dz \tag{9}$$

where $S(z)$ denotes the flow distance from the leading edge to the trailing edge of the differential surface dz and P_{max} is the maximum perimeter of the surface.

These formulas along with the semiempirical results obtained by Yovanovich and Jafarpur (1993) for horizontal surfaces facing upward or downward will be used to determine the composite body-gravity function for cuboids.

DIFFUSIVE LIMIT

The Nusselt number for the diffusive limit is obtained from the dimensionless shape factor $S^{\star}_{\sqrt{A}}$ which is the solution to the classical Dirichlet problem:

$$\nabla^2 \phi(\vec{r}) = 0 \tag{10}$$

at all external points $\vec{r} > \vec{r}_b$, and the dimensionless potential $\phi(\vec{r})$ must satisfy the two Dirichlet boundary conditions:

i) at all points on the body $\vec{r} = \vec{r}_b$,

$$\phi(\vec{r}) = 1 \tag{11}$$

ii) at all remote points in space $\vec{r} \to \infty$,

$$\phi(\vec{r}) \to 0 \tag{12}$$

The dimensionless shape factor is obtained from the following area-mean expression with the characteristic length set to $\mathcal{L} = \sqrt{A}$:

$$S^\star_{\sqrt{A}} = \frac{\sqrt{A}}{A} \iint_A -\nabla\phi \cdot \vec{n} \, dA \tag{13}$$

where $\vec{n}$ is the outward directed normal at all points on the body. The diffusive limit $Nu^\infty_{\sqrt{A}} \equiv Q\sqrt{A}/(k\ A\ (T_0 - T_\infty))$ is equivalent to $S^\star_{\sqrt{A}}$. It has be shown (Yovanovich, 1987a) that there is equivalence between the following dimensionless physical parameters:

$$Q^\star_{\sqrt{A}} = Nu^\infty_{\sqrt{A}} = S^\star_{\sqrt{A}} = C^\star_{\sqrt{A}} = 1/R^\star_{\sqrt{A}} \tag{14}$$

where Q is the total heat flow rate from the isothermal body, C is the capacitance of an isopotential body, and R is the thermal resistance due to pure conduction through the constant property medium surrounding the isothermal body.

Ellipsoid Shape Factor

There are a small number of exact solutions available for certain body shapes such as isothermal ellipsoids, oblate and prolate spheroids, sphere, elliptical and circular disks. Yovanovich (1987a) has shown that the general solution for an isothermal ellipsoid with semiaxes $a \geq b \geq c$ reduces to all of the special cases given above. The dimensionless thermal resistance (Yovanovich, 1987a) is

$$R^\star_{\sqrt{A}} = k\sqrt{A}R \tag{15}$$

$$= \frac{1}{\sqrt{8\pi}} \left[\frac{1}{\tan^2 \varphi} + \frac{v}{\sin\varphi} \left[\frac{F(\varphi, \kappa)}{\tan^2 \varphi} + E(\varphi, \kappa) \right] \right]^{\frac{1}{2}} .$$

$$F\left(\varphi, \frac{\sqrt{1-v^2}}{\sin\varphi}\right) \tag{16}$$

with variables u, v, φ, κ defined with respect to the semiaxes:

$$u = \frac{c}{a}, \quad v = \frac{b}{a}, \quad 1 \geq v \geq u$$

and

$$\varphi = \cos^{-1} u, \quad \kappa^2 = \frac{v^2 - u^2}{v^2(1-u^2)}$$

and $F(\varphi, \kappa)$ and $E(\varphi, \kappa)$ are incomplete elliptic integrals of the first and second kind respectively (Abramowitz and Stegun, 1970). Analytic solutions for bodies which have edges and corners are not available. Therefore approximate values of the shape factors for cuboids (e.g., cube, long and short square parallelepipeds, rectangular and square plates must be computed numerical or estimated from known analytic solutions.

Yovanovich (1987a) has shown that shape factors of bodies which are nominally *similar*, ie. they have i) identical surface areas and ii) similar aspect ratios, will have very nearly the same dimensionless shape factor provided the characteristic length is based on the square-root of the total active surface area. Several special cases useful for this work are given below.

Prolate Spheroid Shape Factor

The dimensionless shape factor of prolate spheroids $a \geq b = c$ is given by

$$S^\star_{\sqrt{A}} = \sqrt{\frac{\pi}{2}}\, 4e \left\{ \left[u^2 + \frac{u}{e} \sin^{-1} e \right]^{\frac{1}{2}} \frac{1}{2} \ln\left(\frac{1+e}{1-e} \right) \right\}^{-1} \tag{17}$$

where $u = c/b$ and $e = \sqrt{1-u^2}$, which is called the eccentricity. The aspect ratio is $AR = 1/u$.

If $AR \geq 5$, the prolate spheroid shape factor can be approximated accurately by the asymptote

$$S^\star_{\sqrt{A}} = \frac{4\sqrt{L/D}}{\ln(2L/D)} \tag{18}$$

with an error $\leq 0.68\%$.

Right Circular Cylinder Shape Factor

The dimensionless shape factor for solid right circular cylinders of length L and diameter D can be obtained from Smythe's (1956, 1962) correlation of his approximate analytical solution of the capacitance C:

$$S^\star_{\sqrt{A}} = \frac{3.1915 + 2.7726(L/D)^{0.76}}{\sqrt{1 + 2(L/D)}} \qquad 0 \leq L/D \leq 8 \tag{19}$$

with a reported maximum error of 0.2%.

Elliptic Disk Shape Factor

The analytical solution for isothermal elliptical disks with major axes: $L = 2a$ and $D = 2b$ with $L/D \geq 1$

can be obtained from the general ellipsoid solution given above. With $a \geq b; c = 0$ the general ellipsoid solution reduces to the elliptic disk solution (Yovanovich 1987a):

$$R^{\star}_{\sqrt{A}} = \frac{1}{S^{\star}_{\sqrt{A}}} = \frac{1}{\sqrt{8\pi}}\sqrt{\frac{b}{a}}\, K\left(\sqrt{1-(b/a)^2}\right) \tag{20}$$

The complete elliptic integral $K(\cdot)$ can be approximated with a maximum error less than 1 % by the following two simple algebraic expressions (Yovanovich 1987a):

$$K\left(\sqrt{1-(b/a)^2}\right) = \frac{2\pi}{\left(1+\sqrt{b/a}\right)^2} \qquad 0.20 < \frac{b}{a} \leq 1 \tag{21}$$

and

$$K\left(\sqrt{1-(b/a)^2}\right) = \ln\left(\frac{4a}{b}\right) \qquad 0 < \frac{b}{a} < 0.20 \tag{22}$$

These approximations of $K(\cdot)$ will be used shortly to develop approximations of shape factors for square and rectangular plates for which analytic solutions are unavailable.

The numerical values of the shape factors for the prolate spheroids, solid right circular cylinders and elliptic disks are compared against each other for a range of the aspect ratio $1 \geq L/D \geq 8$ in Table 1. Clearly the values are nearly the same for large aspect ratio and the differences are relatively small for aspect ratio $L/D = 1$ where the prolate spheroid becomes a sphere and the elliptic disk becomes a circular disk. The maximum difference of approximately 11 % is found between the values of the sphere and the circular disk. The three-dimensional bodies (sphere and right circular cylinder of unit aspect ratio) differ by less than 3 %.

Table 1. Effect of Body Aspect Ratio: Prolate Spheroids, Circular Cylinders and Elliptical Disks

AR L/D	Prolate Spheroids	Right Cylinders	Elliptical Disks
1	3.545	3.443	3.192
2	3.566	3.527	3.288
3	3.628	3.622	3.434
4	3.706	3.714	3.579
5	3.790	3.803	3.716
6	3.875	3.887	3.845
7	3.959	3.965	3.952
8	4.040	4.040	4.080

Square Cuboid Shape Factors

Table 2 shows that the numerical values of dimensionless shape factor for the square cuboids, $H = W; L/H \geq 1$, lie approximately 5 % below the values for the right circular cylinder and the prolate spheroids provided the bodies have identical surface areas and have similar aspect ratios. Therefore, the solutions for the right circular cylinder and the prolate spheroid may be used to estimate the shape factors for the square cuboids. The asymptote for the prolate may be used to estimate the dimensionless shape factor for all square cuboids with $L/W \geq 8$.

Table 2. Effect of Body Aspect Ratio: Prolate Spheroids, Circular Cylinders and Square Cuboids

AR L/D	Prolate Spheroids	Right Cylinders	Square Cuboids
1	3.545	3.443	3.373
2	3.566	3.527	3.406
3	3.628	3.622	3.465
4	3.706	3.714	3.532
5	3.790	3.803	3.598

Rectangular and Square Plate Shape Factors

The very close agreement between the analytic values of the shape factors of elliptic disks and the numerical values of the shape factors for rectangular plates which are similar (equal surface areas and equal aspect ratios) permits one to use the approximate expressions given above for elliptic disks to develop the following approximations for rectangular plates of length L and width W and square plates $L = W$:

$$S^{\star}_{\sqrt{A}} = 0.8\,\frac{\left(1+\sqrt{L/W}\right)^2}{\sqrt{L/W}} \qquad 1 \leq \frac{L}{W} \leq 5 \tag{23}$$

and

$$S^{\star}_{\sqrt{A}} = \frac{\sqrt{8\pi L/W}}{\ln\left(4L/W\right)} \qquad 5 < \frac{L}{W} < \infty \tag{24}$$

General Cuboid Approximate Shape Factor

Ellipsoid Method

The dimensionless shape factor for the general cuboid, $[H, W, L]$, is more difficult to estimate accurately because numerical values are presently unavailable for comparison with the general ellipsoid solution. Because of the close agreement of the dimensionless shape factors for the rectangular and square plates with the exact values of the dimensionless shape factors for elliptic disks and circular disk; and the acceptable agreement of the dimensionless shape factors for the square cuboids against the exact values for the dimensionless shape factor for the prolate spheroids, and the very accurate values for the right circular cylinder, it is proposed that the general ellipsoid solution can be used to estimate the dimensionless shape factors for arbitrary cuboids.

The approximate dimensionless shape factor for a general cuboid $[H, W, L]$ is estimated by

$$S^\star_{\sqrt{A}}\{Cuboid[H, W, L]\} = 0.975 S^\star_{\sqrt{A}}\{Ellipsoid[a, b, c]\} \tag{25}$$

where $2a, 2b$, and $2c$ are set equal to the largest, middle and smallest values of $[H, W, L]$ respectively. When $2c = 0$, the cuboid becomes a rectangular plate, then the factor 0.975 should be omitted from the above relationship to provide better agreement with the known rectangular plate numerical results. This method of estimating the dimensionless shape factor is expected to be accurate to at least 3 %. It is, however, somewhat computationally intensive because three incomplete elliptic integrals must be computed accurately. Therefore alternate methods will be proposed below.

Circular Cylinder: Inscribing and Circumscribing Method

To minimize the computational effort the following less accurate method is proposed for estimating dimensionless shape factors for arbitrary cuboids.

Let the largest, middle and smallest values of the cuboid $[H, W, L]$ be denoted: L_{max}, L_{mid} and L_{min} respectively.

Assume that the cuboid is inscribed by a circular cylinder whose length is equal to L_{max} and whose diameter is equal to L_{min}. Use Eq. (19) to obtain the first estimate of the cuboid shape factor. Next circumscribe a circular cylinder about the cuboid such that the cylinder length is equal to L_{max} as before; but now the diameter of the circumscribing cylinder will be equal to $\sqrt{L_{mid}^2 + L_{min}^2}$. Obtain a second estimate of the dimensionless shape factor by means of Eq. (19). Take the geometric mean of the two estimates as an approximation of the of the dimensionless shape factor of the given cuboid. This approach provides a quick, reasonably accurate, estimate.

Circular Cylinder: Aspect Ratio Method

Another approach is to obtain the geometric mean value of the length-to-diameter ratios from the two calculations above which gives two aspect ratios which are defined as the maximum and minimum aspect ratios of the cuboid:

$$AR_{max} = \frac{L_{max}}{\sqrt{2L_{min}\sqrt{L_{min}^2 + L_{mid}^2}}} \tag{26}$$

and

$$AR_{min} = \frac{L_{min}}{\sqrt{2L_{mid}\sqrt{L_{mid}^2 + L_{max}^2}}} \tag{27}$$

The factor 2 was introduced into the above expressions to improve the accuracy of the estimates of Eq. (19) for the special case of a cube, $H = W = L = 1$.

This approach is also quick and it should give results with acceptable accuracy; the maximum errors should be less than approximately 3 %. For example, consider the cuboid $H = 1, W = 2, L = 3$. The minimum and maximum aspect ratios are 0.263 and 1.418 respectively, and the corresponding dimensionless shape factors calculated with Eq. (19) are 3.397 and 3.476 respectively. The difference is only 2.3 % and the average value is 3.436.

If we take the geometric mean of the minimum and maximum aspect ratios we obtain $AR = 0.611$ which with Eq. (19) gives $S^\star_{\sqrt{A}} = 3.420$ as another estimate of the cuboid shape factor. The two estimates differ by less than approximately 0.5 %, and therefore the expected value is very close to 3.43.

This approach is a fast and relatively accurate method of estimating $S^\star_{\sqrt{A}}$ because the shape factor has a relatively weak dependence on the shape and aspect ratio of the body as noted above.

BODY GRAVITY FUNCTION FOR VERTICAL AND HORIZONTAL CUBOIDS

Vertical and horizontal cuboids are characterized by the fact that they possess vertical and horizontal surfaces with respect to the gravity vector, Fig. 1. The buoyancy-induced fluid flow patterns over these surfaces are complex and therefore extemely difficult to model accurately.

To appreciate the development of the body-gravity functions for vertical and horizontal cuboids, the discussion will be restricted to a thin laminar hydrodynamic boundary layer $Ra_{\sqrt{A}} > 10^4$ with a much thinner thermal boundary layer $Pr >> 1$ embedded within it.

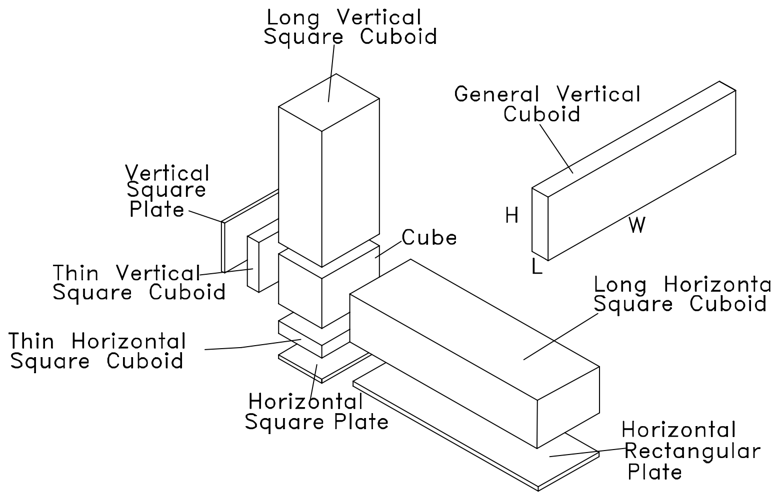

Figure 1: Schematic of Several Horizontal and Vertical Cuboids

For example consider two cases: 1) vertical square prisms and 2) long horizontal square prisms (see Fig. 1). In the first case the fluid flow commences at the lower stagnation point (located at the centroid) of the bottom surface; the fluid flows over the bottom surface in some complex flow pattern; it traverses the lower perimeter; then it flows along the sides and ends to the perimeter of the top horizontal surface. If the fluid velocity is sufficiently small, the fluid will traverse the top perimeter and flow over the top surface from the perimeter towards the top centroid where the fluid leaves the top surface in a plume.

In this case the buoyancy-induced flow is said to be in the *series flow arrangement*; the total body surface is decomposed into i) the bottom surface, ii) the side and end surfaces together, and iii) the top surface.

The body-gravity functions for the respective surfaces are obtained by means of Eqs. (6) and (8) for the series flow arrangement described above. Since the horizontal surfaces cannot be handled by Eq. (5); they are modelled by means of the semiempirical models developed by Yovanovich and Jafarpur, 1993.

Alternate flow patterns and arrangement of surfaces are discussed in great detail in the study of Jafarpur, 1992.

One objective of this study is to present a simple model which captures the essential physics of buoyancy-induced flow over the surfaces of horizontal cuboids and provides relatively accurate results over a wide range of Rayleigh and Prandtl numbers.

General Body-Gravity Function For Cuboids

The semiempirical results for rectangular surfaces of area $A = LW$ facing upward or downward are (Yovanovich and Jafarpur, 1993):

$$G_{\sqrt{A}}^{top} = 2^{1/8} \left(\frac{L}{W}\right)^{1/8} \qquad \text{and}$$

$$G_{\sqrt{A}}^{bot} = \frac{1}{2}\, G_{\sqrt{A}}^{top} \qquad \frac{L}{W} \geq 1 \tag{28}$$

They are combined with the formula applicable for vertical side and end surfaces:

$$G_{\sqrt{A}}^{sidends} = \left(\frac{P}{\sqrt{A}}\right)^{1/4} \tag{29}$$

where the cuboid perimeter perpendicular to the gravity vector is $P = 2(L + W)$, and the total area of the sides plus ends is $A = 2(HW + HL)$. Assuming series fluid flow over the bottom, side plus ends combined, and the top surfaces according to Eq. (8) gives

$$G_{\sqrt{A}}^{cuboid} = 2^{1/8} \left[\frac{0.625\ L^{4/3} W + H(L + W)^{4/3}}{(HW + HL + LW)^{7/6}}\right]^{3/4} \tag{30}$$

The first term in the numerator of the above expression corresponds to the contribution of the top and bottom surfaces and the second term corresponds to the contribution of the sides and ends to the overall body gravity function.

The denominator is associated with the total surface area of the cuboid.

Body-Gravity Functions For Several Special Cases

Several special cases arise from the above general expression, Eq. (30). These will be discussed next.

Horizontal Rectangular Plates: $H = 0, L \geq W$

$$G_{\sqrt{A}}^{rect\ plate} = 0.7665 \left(\frac{L}{W}\right)^{1/8} \qquad \frac{L}{W} \geq 1 \tag{31}$$

Horizontal Square Plate; Both Sides Active: $H = 0, W = L$

$$G_{\sqrt{A}}^{sq\ plate} = 0.7665 \tag{32}$$

Vertical Rectangular Plates; Both Sides Active: $L = 0$

$$G_{\sqrt{A}}^{vert\ plate} = 2^{1/8} \left(\frac{W}{H}\right)^{1/8} \qquad 0 < \frac{W}{H} < \infty \tag{33}$$

If the vertical plate has one side active only, omit the factor $2^{1/8}$.

Long Horizontal Square Prisms; Active Ends: $H = W << L$

$$G_{\sqrt{A}}^{long\ prism} = 0.856 \left(\frac{L}{H}\right)^{1/8} \qquad \frac{L}{H} > 10 \tag{34}$$

Vertical Square Prisms With Active Ends: $L = W, 0 \leq H/W < \infty$

The general expression, Eq. (30), reduces to

$$G_{\sqrt{A}}^{vert\ sq\ prism} = 2^{1/4}\frac{(0.250 + H/W)^{3/4}}{(0.500 + H/W)^{7/8}} \tag{35}$$

The constant in the numerator has been rounded to the value 0.250 for convenience; this gives for the zero thickness prism, $H/W = 0$, a value of the body gravity function approximately 0.6 % greater than the value given above.

Horizontal Cube: $H = W = L = 1$

$$G_{\sqrt{A}}^{cube} = 0.984 \tag{36}$$

The validity and accuracy of the general expression, Eq. (30), and some of the specific expressions for the body-gravity functions for horizontal cuboids will be verified against air data in the following section over a wide range of the Rayleigh number.

COMPARISON OF CUBOID MODEL WITH AIR DATA

The proposed cuboid model(s) is compared against extensive air data ($Pr = 0.71$) (Chamberlain, 1983; Clemes, 1990; Hassani, 1987; Karagiozis, 1991; and Saunders, 1936) for several special cases of cuboids over a wide range of Rayleigh numbers: $10^3 < Ra_{\sqrt{A}} < 10^{11}$. The close agreement between the cuboid model and the air data are clearly seen in graphical form in Figs. 2 through 7.

The cuboids with the estimated diffusive limits and body-gravity functions are given in Table 3. Although the three-dimensional bodies tested varied significantly in shape, aspect ratio and orientation, the calculated diffusive limits were found to lie in the range: $3.28 \leq Nu_{\sqrt{A}}^{\infty} \leq 4.10$. The difference between the smallest values corresponding to the $[1, 1, .1]$ and $[.1, 1, 1]$ cuboids (thin square disks) and the largest value corresponding to the long square prism $[1, 1, 10.13]$ differ by approximately 25 %.

The calculated body-gravity functions were found to lie in the range: $0.841 \leq G_{\sqrt{A}} \leq 1.20$. The smallest value corresponds to the horizontal thin cuboid $[.1, 1, 1]$ as expected, and the largest value corresponds to the long horizontal square cuboid $[1, 1, 10.13]$. The difference between the largest and smallest values is approximately 43 %.

The RMS and maximum percent differences between the cuboid model predictions and the measured values of $Nu_{\sqrt{A}}$ are presented in Table 3. The RMS % differences lie in the range: $2.5 - 6.1$, and the maximum percent differences lie in the range: $3.8 - 12.1$.

The agreement in general is seen to be very good to excellent. The maximum differences between the data and the model occur at the lowest Rayleigh number values for the cuboid $[1, 1, .1]$ shown in Fig. 4 and the last $6 - 9$ data points of the cuboid $[1, 1, 10.13]$ at the highest Rayleigh numbers shown in Fig. 7.

The excellent agreement between the data of cuboids: $[1, 1.43, .064]$ and $[1, .666, .024]$ and the proposed cuboid model seen in Figs. 5 and 6 refutes the relatively large difference observed in Fig. 4.

The agreement between the present cuboid model and the air data of Saunders (1936) shown in Fig. 6 is very good, especially at the lower end of the Rayleigh number range. The Saunders cuboid is a vertical rectangular plate. The RMS and maximum percent differences are not given in Table 3 because the data were obtained by digitization of points presented in graphical form.

Jafarpur (1992) has demonstrated that the proposed cuboid model can be modified to provide an upper bound on the body-gravity function by assuming that

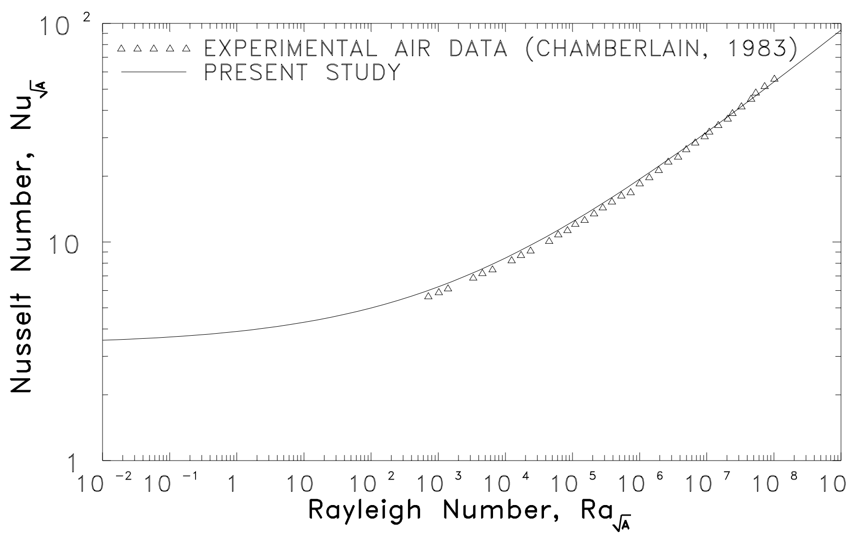

Figure 2: Comparison of Present Cuboid Model with Air Data for Cube (H = W = L = 1).

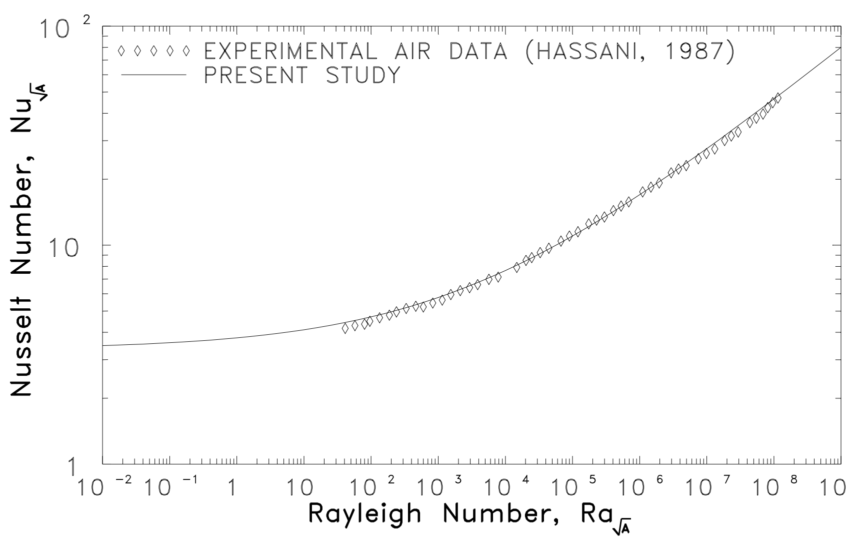

Figure 3: Comparison of Present Cuboid Model with Air Data for Horizontal Square Disk (H = 0.1, W = L = 1).

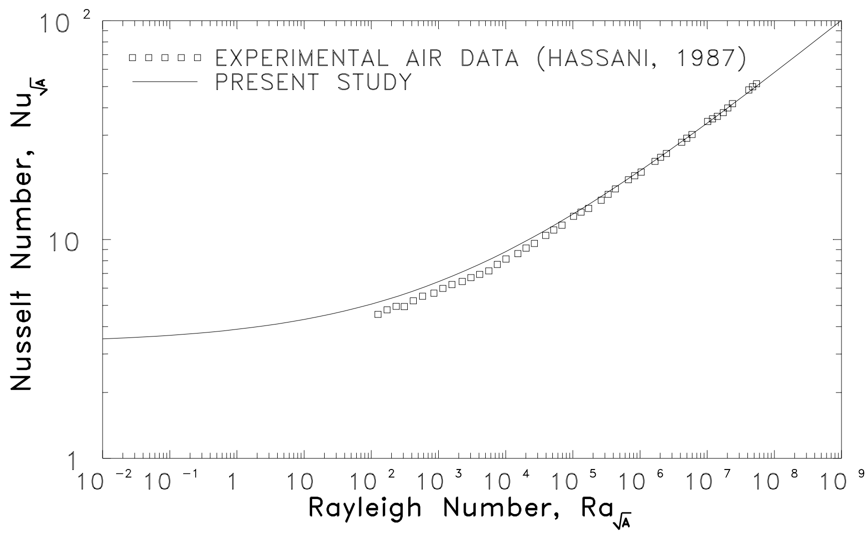

Figure 4: Comparison of Present Cuboid Model with Air Data for Vertical Square Disk (H = W = 1.0, L = 0.1).

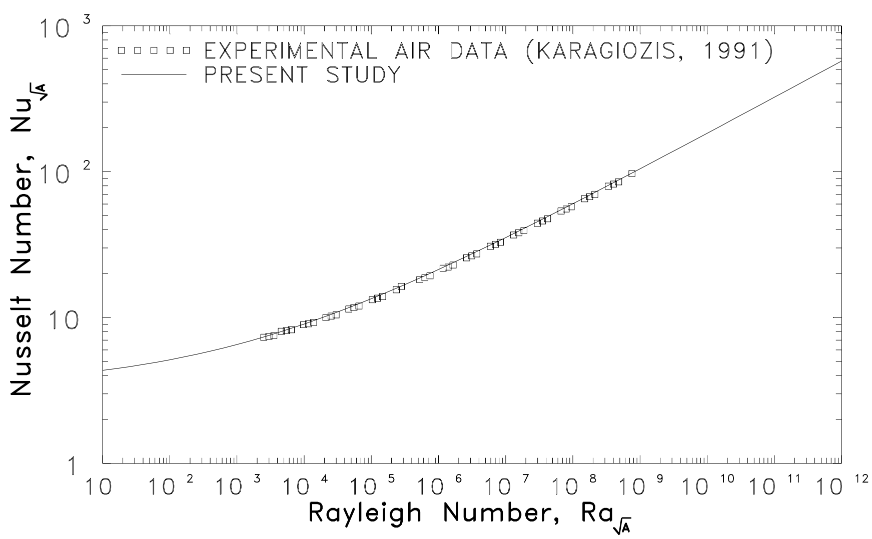

Figure 5: Comparison of Present Cuboid Model with Air Data for Vertical Rectangular Plate (H = 1, W = 1.43, L = 0.064).

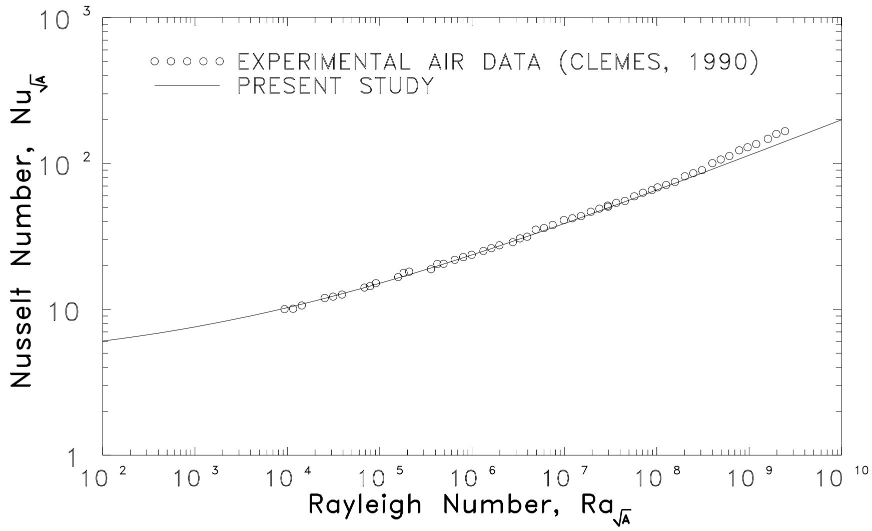

Figure 6: Comparison of Present Cuboid Model with Air Data for Long Horizontal Square Prism (H = W = 1.0, L = 10.13).

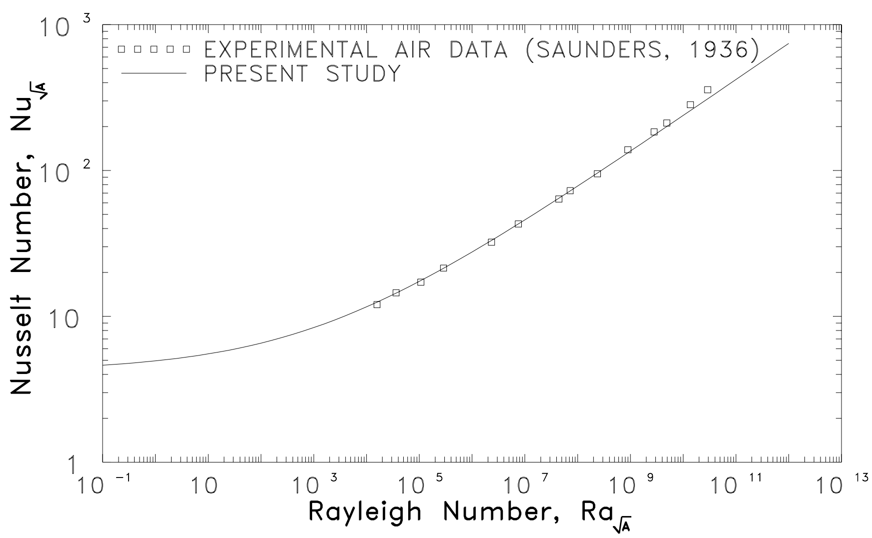

Figure 7: Comparison of Present Cuboid Model with Air Data for Vertical Rectangular Plate (H = 1, W = 3.03, L = 3.3×10^{-5}).

the bottom, sides and ends, and the top surfaces are in the *parallel flow arrangement* to reflect the modified fluid flow patterns over these surfaces. This approach provides an upper bound for the cuboid model. The data at the highest Rayleigh number range appear to approach the upper bound predictions.

Chang et al. (1988) reported nine air data points for the horizontal cuboid $[1, 1, 12.4]$ in the narrow Rayleigh number range between 10^7 to 10^8. From the proposed cuboid model one calculates: $Nu^\infty_{\sqrt{A}} = 4.37$ and $G_{\sqrt{A}} = 1.188$. The data lie within 7.5 % of the model predictions.

Table 3. Cuboids, Diffusive Limits, Body-Gravity Functions, RMS and Maximum Percent Differences for Bodies Used in Comparison

Cuboid $[H, W, L]$	Diffusive Limit $Nu^\infty_{\sqrt{A}}$	Body Gravity Function $G_{\sqrt{A}}$	RMS % Diff.	MAX % Diff.
$[1, 1, 1]$	3.39	0.984	3.8	6.5
$[.1, 1, 1]$	3.34	0.841	5.9	11.8
$[1, 1, .1]$	3.34	1.064	6.1	12.1
$[1, 1.43, .064]$	3.33	1.11	2.5	3.8
$[1, 3.03, \simeq 0]$	3.44	1.25	–	–
$[1, 1, 10.13]$	4.10	1.20	4.7	11.9

CONCLUSIONS

A general cuboid model has been proposed for natural convection heat transfer from horizontal and vertical isothermal cuboids of arbitrary shape and orientation for a wide range of Rayleigh and Prandtl numbers. Methods for estimating the dimensionless diffusive limit and the body-gravity function are presented. Several expressions for the diffusive limit and the body-gravity function respectively are presented for some special cases.

The proposed model which is based on the linear superposition of the diffusive limit and the boundary layer asymptote is compared against experimental data obtained from natural convection heat transfer from isothermal cuboids into a sufficiently large volume of stagnant air over a wide range of Rayleigh number.

The tested cuboids ranged from horizontal and vertical *thin* square disks, vertical *thin* rectangular plates, a horizontal cube, and two long horizontal square prisms.

The agreement between the proposed cuboid model and the data is very good for the range of bodies tested over the Rayleigh number range: $10^3 \leq Ra_{\sqrt{A}} \leq 10^{11}$. RMS and maximum percent differences are reported.

The proposed cuboid model is relatively simple, easy to use and quite accurate, and can be modified easily to model heatsinks.

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