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Linearization Method for Buoyancy Induced Flow over a Nonisothermal Vertical Plate

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A simple model is developed to predict natural convective heat transfer from an isolated vertical plate with an arbitrary surface temperature variation. The concept of linearized approximation is applied to the boundary-layer equations with the use of an effective velocity. The effective velocity characterizes the buoyancy induced flow in the boundary layer, and it is determined by relating the total work done by the buoyant force on the fluid to the kinetic energy of the fluid flow. The temperature distributions in the fluid and the surface heat-flux variations are predicted for the cases in air with a variety of different surface temperature conditions. The present solutions are in excellent agreement with existing solutions that are obtained by using more rigorous solution techniques.

Nomenclature

a, b = constants
C = coefficient given by Eq. (24)
C_1, C_2 = coefficients
f_a = function defined by Eq. (15)
Gr_a = Grashof number. $g\beta T_a x_a^3/\nu^2$
Gr_x = Grashof number. $g\beta T_w x^3/\nu^2$
$\bar{Gr}_x$ = Grashof number. $g\beta \bar{T}_w x^3/\nu^2$
g = gravitational acceleration
k = thermal conductivity
Nu_a = Nusselt number. $q_w x_a/T_a k$
Nu_x = Nusselt number. $q_w x/T_w k$
Pr = Prandtl number. ν/α
q = local heat flux
q^* = dimensionless heat flux. q/q_{w_0}
Re_x = Reynolds number. $u_e x/\nu$
T = temperature excess
$\bar{T}_w$ = average wall temperature
t = variable defined by Eq. (9)
u = local velocity in x direction
u_e = effective velocity
v = local velocity in y direction
x = vertical coordinate along the plate
y = horizontal coordinate normal to the plate
α = thermal diffusivity of fluid
β = thermal expansion coefficient
ΔT_{w_i} = temperature difference, $T_{w_i} - T_{w_{i-1}}$
δ = boundary-layer thickness
ζ = dummy variable
η = similarity variable
θ = dimensionless temperature. T/T_{w_0}
ν = kinematic viscosity
ξ = dimensionless x coordinate. x/x_1
χ = delay factor

Subscripts

a = arbitrary references
i = parameters at ith step

w = wall conditions
0 = parameters at leading section

Introduction

THE study of natural convection has been carried out over the past several decades by many investigators. To the authors' present knowledge. however. there exists no simple analytical solution for predicting the heat transfer characteristics of a vertical plate with an arbitrary temperature variation prescribed along the surface. Existing analytical approaches. especially those which attempt to obtain exact solutions to the boundary-layer form of the conservation equations. are limited to a few specific cases with well-defined surface conditions. Uniform surface heat flux.[1] temperature variations of the power and exponential form.[2] and a line source on an adiabatic plate[3] are the only surface thermal conditions that allow similarity transformations for problems involving vertical plates. The similarity transformations reduce the boundary-layer equations to a set of ordinary differential equations which, in turn. must be solved numerically.

In order to expand available solutions to include problems with nonsimilar surface conditions. numerous researchers have conducted investigations using various methods and techniques. The problem with a step change in surface temperature was first examined experimentally by Schetz and Eichhorn.[4] Their work was followed by Hayday et al.[5] and Sokovishin and Érman[6] who used numerical methods. and by Smith.[7] Kelleher.[8] and Kao[9] who used series expansions on the same problem. More recently. Lee and Yovanovich.[10,11] and Park and Tien[12] also developed approximate models for a vertical plate with changes in thermal conditions.

All existing solution techniques for solving nonsimilar problems with general thermal conditions at the surface are of an approximate nature and can be categorized into several classes. The class of fully numerical methods. either finite-volume. finite-difference. or finite-element methods. is the most versatile for handling general boundary conditions. and is capable of providing close-to-exact solutions. The second class consists of series expansion methods. By applying appropriate coordinate transformations and the Merk-type series expansions. Yang and his co-workers[13] reduced the boundary-layer equations to sets of ordinary differential equations which were solved numerically for a vertical plate with prescribed surface temperature and heat-flux variations. Others include the methods of local similarity and local nonsimilarity. These methods were developed by Kao et al.[14] for cases with nonuniform surface thermal conditions.

As with most analytical approaches. however. applicability of the aforementioned approximate methods cannot be ex-

tended to include problems with general boundary conditions without losing the level of confidence in the solution accuracy that may have been acquired by comparing the results with others for some specific cases. This was indicated by 1) the inability of the perturbation method of Kao[9] in predicting the surface heat-flux distribution for the case involving a step change in temperature when the downstream surface temperature is prescribed equal or close to the ambient fluid temperature; 2) the premature overshooting of Kelleher's asymptotic series solutions[8] over the limiting values at downstream locations for the cases with a step change in surface temperature; and 3) the divergence observed in the local similarity and local nonsimilarity solutions of Kao et al.[14] when linearly decreasing heat flux and sinusoidal temperature variations are specified along the surface. Furthermore, in most situations, these limitations are not foreseen until the resulting solutions are compared case-by-case with more reliable solutions. As the surface condition becomes arbitrary it is found that the existing analytical methods either fail to become applicable or become so complex that it is ultimately simpler to seek solutions by direct numerical integration of the governing equations.

Simple, easy-to-use solutions are of a great value to those involved in applications where most problems arise in such a way that it is necessary to obtain quick solutions at a lower cost. The objective of the current study is to provide a simple expression for predicting heat transfer results along a vertical surface with arbitrarily prescribed temperature variations. A new approximate method is developed based on a linearization of the conservation equations. The linearization is carried out by introducing an effective velocity which characterizes the boundary-layer flow induced by the buoyant force. The resulting equations are solved and the effective velocity is determined by finding the relationship that exists between the total work done by the buoyant force on the fluid and the effective kinetic energy of the fluid flow. The effective velocity determined in this manner is found to be analogous to the freestream velocity of an externally induced flow, thereby allowing the present heat transfer analysis to proceed in such a way that is similar to the forced convection analysis. A simple, explicit solution, which only requires an algebraic sum of a finite number of power terms, is obtained for local heat-flux distributions.

Problem Statement

The geometric configuration and the coordinate system of the present problem are shown in Fig. 1a, where a vertical plate is depicted with an arbitrary temperature variation prescribed at the surface. Also shown in the right side of Fig. 1a is the growth of the hydrodynamic boundary-layer δ. The thickness of the boundary layer is shown greatly exaggerated

in the y direction. The plate is suspended in an extensive, quiescent fluid which is assumed to be maintained at uniform temperature. The velocity and temperature fields in two-dimensional, steady-state laminar natural convective flow may be described by the set of usual boundary-layer equations, given as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + g\beta T \tag{2}$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{3}$$

with boundary conditions

$$\text{at} \quad y = 0, \quad u = v = 0, \quad T = T_w(x)$$
$$\text{as} \quad y \rightarrow \infty, \quad u \rightarrow 0, \quad T \rightarrow 0 \tag{4}$$
$$\text{at} \quad x = 0, \quad u = T = 0$$

where T denotes the temperature excess above the ambient fluid temperature. Only the cases where the surface temperature excess is maintained positive will be considered in the present study. This condition ensures that the buoyant force is in the positive x direction everywhere in the fluid, thereby eliminating the possibility of an occurrence of a counterflow.

Without a loss of generality, the surface temperature variation $T_w(x)$ can be discretized into piece-wise uniform step changes as presented in Fig. 1b where only a few steps are shown for brevity. The prescribed temperature condition at the plate surface given in Eq. (4) becomes

$$T(y = 0) = T_{wi} \quad \text{for} \quad x_i \leq x < x_{i+1}; \quad i = 0, 1, 2, \ldots \tag{5}$$

where $x_0 = 0$, and T_{wi} is the average surface temperature over the ith section between $x = x_i$ and x_{i+1}. Obviously, a finer discretization would yield a closer representation of the continuous variation, and in such cases the temperature at the midlocation of the section can be used for T_{wi}. The primary boundary-layer δ that is developed due to the initial step increase of T_{wi} at the leading edge of the plate, is denoted with a subscript 0. In addition to the primary boundary layer, Fib. 1b shows the outer edges of the boundary layers developed at the onset of individual step changes. These inner layers are denoted with the subscripts that correspond to those of the step changes.
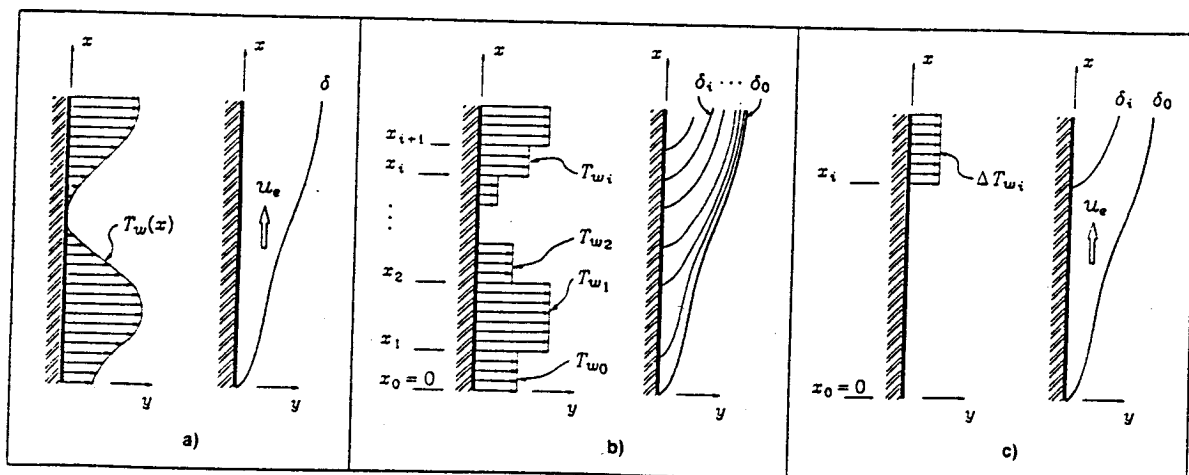

Fig. 1 Geometric configuration and coordinate system shown with schematic representations of surface temperature variation and development of boundary layers where surface temperature is prescribed by a) continuous variation, b) step changes, and c) an isolated step change in the wake of a buoyancy driven flow.

Although the inner layers grow within and ultimately blend into the outer layers, each inner sublayer may be conceptually isolated from the primary boundary layer. Figure 1c exhibits a single step jump in surface temperature at $x = x_1$. The right side of the figure shows the corresponding boundary-layer growth within the wake of the primary, hydrodynamic boundary-layer flow. This case will be discussed further in a later section.

Linearization of Equations

The momentum and energy equations, Eqs. (2) and (3), are mutually coupled and nonlinear: the conditions that make natural convection problems more difficult than forced convection problems, in general, by precluding the use of the method of superposition in obtaining the solution. The equations are linearized and, simultaneously, the energy equation is decoupled from the momentum equation by approximating the convective operator appearing in the left side of Eqs. (2) and (3) as follows:

$$u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \approx u_e(x) \frac{\partial}{\partial x} \qquad (6)$$

where $u_e(x)$ is the effective velocity, defined to be uniform across the boundary layer.

The definition of $u_e(x)$ that is exact in the integral form of the momentum equation, can be obtained by integrating Eq. (6) from $y = 0$ to δ with u as an operand. It follows that

$$u_e(x) = \frac{\dfrac{d}{dx} \displaystyle\int_0^\delta u^2 \, dy}{\dfrac{d}{dx} \displaystyle\int_0^\delta u \, dy} \qquad (7)$$

Equation (1) was used in simplifying the numerator of the above definition. Instead of u, T could have been used arriving at another definition for $u_e(x)$. This definition based on T yields an expression for $u_e(x)$ which is exact in the integral form of the energy equation and is functionally identical to the one obtained from Eq. (7). Either definition can be used in the following analysis in determining $u_e(x)$.

The linearized convective operator on the right side of Eq. (6) can be transformed further by introducing a new variable t such that

$$dt = \frac{dx}{u_e(x)} \qquad (8)$$

or

$$t = \int_0^x \frac{d\zeta}{u_e(\zeta)} \qquad (9)$$

The parameter t can be viewed as the effective residence time of the fluid flow from the leading edge of the plate to the downstream location x.

With the use of Eqs. (6) and (8), the momentum and energy equations in the x-y plane can be transformed into the t-y plane, written as

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} + g\beta T \qquad (10)$$

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2} \qquad (11)$$

and the thermal boundary condition given by Eq. (5) becomes

$$T(y = 0) = T_{w_i} \quad \text{for} \quad t_i \leq t < t_{i+1}: \qquad i = 0, 1, 2, \ldots \qquad (12)$$

where t_i corresponds to x_i, and $t_0 = 0$.

Analysis

The exact solutions to the above linearized equations are found by means of similarity transformations and the method of superposition.[15] They can be written in terms of the complementary error functions for $x_\kappa \leq x < x_{\kappa+1}$ as

$$T = \sum_{i=0}^{\kappa} \Delta T_{w_i} \, \text{erfc} \, \eta_i \qquad (13)$$

$$u = 2g\beta \sum_{i=0}^{\kappa} \Delta T_{w_i}(t - t_i) f_u(\eta_i) \qquad (14)$$

where

$$f_u(\eta) = \begin{cases} \eta \, \text{ierfc} \, \eta & \text{for} \quad \text{Pr} = 1 \\[2mm] \dfrac{2}{1 - \text{Pr}} \left(i^2\text{erfc} \, \eta - i^2\text{erfc} \, \dfrac{\eta}{\sqrt{\text{Pr}}} \right) & \text{for} \quad \text{Pr} \neq 1 \end{cases} \qquad (15)$$

with the similarity variable η_i given by

$$\eta_i = [y/2\sqrt{\alpha(t - t_i)}] \qquad (16)$$

Here, and throughout this article, the parameters with subscript $(i - 1)$ vanish when $i = 0$. The only unknown in the above solutions is the transient variable $(t - t_i)$. Determination of $(t - t_i)$ in terms of x will be carried out in two stages.

First, let us examine the case of flow over an isothermal surface. Although exact similarity solutions are available,[2] it is useful to reexamine the isothermal case as it will provide the basis upon which the current analysis can be extended to include the cases with arbitrary temperature distributions at the surface.

An isothermal case corresponds to the present problem with $\kappa = 0$. Substituting the first term of Eq. (14) into Eq. (7) for u and evaluating the integrals with $\delta \to \infty$ leads to

$$u_e = C_1 g\beta T_{w_0} t \qquad (17)$$

where C_1 is a function of the Prandtl number. Determination of this function is not necessary, and it is sufficient to note that this coefficient is a constant for a given fluid. By differentiating the above equation with respect to t and substituting Eq. (8) for dt, it can be shown that

$$\frac{du_e^2}{dx} = 2C_1 g\beta T_{w_0} \qquad (18)$$

This indicates that the change in the effective kinetic energy of the flow in the x direction is proportional to the buoyant force exerted on the fluid adjacent to the surface. Integration with respect to x gives

$$u_e(x) = \sqrt{2C_1 g\beta T_{w_0} x} \qquad (19)$$

Substituting this into Eq. (9) and evaluating the integral, one finds that

$$t = \frac{x}{u_e/2} \qquad (20)$$

Therefore, the temperature solution given by Eq. (13) can now be expressed in terms of the known parameters, except the value of C_1.

From the temperature solution the surface heat flux $q_w(x)$ can be obtained by employing Fourier's law of conduction at the plate surface. After using Eq. (20) for t and rearranging, the local Nusselt number may be written in the form which

is identical to the one obtained in laminar forced convection analysis

$$Nu_x = C_2 Re_x^{1/2} \qquad (21)$$

where C_2 is another function of the Prandtl number and Re_x denotes the local Reynolds number defined as

$$Re_x = (u_e x/\nu) \qquad (22)$$

Equation (21) reveals the similarity between natural and forced convective heat transfer analyses. It indicates that the buoyancy induced effective velocity $u_e(x)$ can be seen to be analogous to the externally induced freestream velocity of the forced convection study. Substituting Eq. (19) for $u_e(x)$, Eq. (21) becomes

$$Nu_x = CGr_x^{1/4} \qquad (23)$$

which is identical to the form of known laminar natural convection results.

The coefficient C consists of C_1 and C_2. It can be determined either by conserving the momentum and energy over the boundary layer,[10] or simply by comparing the heat transfer result with known data. Since the prime objective of the present investigation is to seek a simple solution that requires the least computational effort the latter will be employed to determine C. For example, by comparing the above result with the correlation equation proposed by Churchill and Usagi,[16] it can be shown that

$$C = \frac{0.503 Pr^{1/4}}{[1 + (0.492/Pr)^{9/16}]^{4/9}} \qquad (24)$$

With the use of Eqs. (19), (20), and (24), the parameters associated with the temperature solution for the isothermal case are completely determined, and the similarity variable η_0 can be written as

$$\eta_0 = \frac{C\sqrt{\pi}}{2} Gr_x^{1/4} \frac{y}{x} \qquad (25)$$

This concludes the first stage of the analysis in determining the transient variables. The analysis so far has shown that it is possible to derive the proper heat transfer relationship for isothermal cases from the linearized differential equations. We shall now extend the above analysis to determine $(t - t_i)$ for problems involving multiple step changes in surface temperature.

Recall that in linearizing the governing differential equations, the effective velocity $u_e(x)$ was introduced as an input parameter to the differential equations and, therefore, it needs to be determined based on information external to the equations. In the above example with an isothermal surface, this was accomplished by equating the integral forms of the equations. It resulted in Eq. (18) which was subsequently integrated to obtain Eq. (19) for $u_e(x)$. Similarly, by assuming that the proportional relationship observed from Eq. (18) between the change in kinetic energy of the fluid flow and the buoyant force on the fluid is not affected by the existence of the variable temperature condition at the surface, the effective velocity $u_e(x)$ for the case with an arbitrarily varying $T_w(x)$ is obtained by integrating Eq. (18) with $T_w(x)$ in place of T_{w0}. It results in

$$u_e(x) = \sqrt{2C_1 g\beta \bar{T}_w x} \qquad (26)$$

where $\bar{T}_w$ is defined as

$$\bar{T}_w = \frac{1}{x} \int_0^x T_w(\zeta) \, d\zeta \qquad (27)$$

As was previously observed in the case of an isothermal plate, this velocity given by Eq. (26) can be seen to be anal-

ogous to the freestream velocity of the forced convection analysis. In the following, the effects that each isolated step jump in surface temperature has on the overall temperature solution will be determined.

Consider the fluid flow over a vertical plate which experiences a single step jump in temperature ΔT_w, at some downstream location $x = x_i$. This is a classical problem that appears in the study of forced convective heat transfer from a flat plate that has an unheated starting length upstream of the heated section as shown in Fig. 1c. The hydrodynamic boundary-layer growth begins at the leading edge whereas the thermal boundary-layer development begins at $x = x_i$. Through use of an integral boundary-layer solution[17] it can be shown that χ, which accounts for the unheated starting-length effects on the heat transfer, can be obtained as

$$\chi \equiv \frac{Nu_x}{Nu_x(x_i \to 0)} = \left[1 - \left(\frac{x_i}{x} \right)^a \right]^{-b} \qquad (28)$$

with $a = \frac{3}{4}$ when the approaching flow velocity $u_e(x)$ is given by Eq. (26), and $b = \frac{1}{3}$ for the externally induced laminar flow of a high Prandtl number fluid. Here, the effective velocity for a nonisothermal surface case is assumed to maintain the same variation with x as in the isothermal case.

The above thermal delay factor is derived providing that the entire boundary-layer flow over the heated section is solely due to the externally induced flow described by Eq. (26). Therefore, the effects that the buoyant force has on the fluid flow beyond $x = x_i$ are not accounted for. Since the value of b depends, in part, on the type of flow, it is seen that b is the parameter which can be modified to account for the buoyant effects on the flow. In this regard, the limiting values of the Prandtl number and the definition of the parameter t given by Eq. (9) were examined, and it is found that the value of b is bounded and it depends on the Prandtl number. An examination of heat transfer results shows that for air ($Pr \approx 0.7$), $b = \frac{1}{3} + \frac{1}{8}$ results in the best agreement with existing data obtained by various techniques for different temperature distributions.

Note that in the above definition of the delay factor, Nu_x denotes the local Nusselt number corresponding to the ith step change in surface temperature and $Nu_x(x_i \to 0)$ is the local Nusselt number corresponding to the isothermal case. Also, from the temperature solution given by Eq. (13), the local Nusselt number can be found to be inversely proportional to the square root of the transient variable. Hence, in terms of the transient variables, χ can be expressed also as

$$\chi \equiv \frac{Nu_x}{Nu_x(x_i \to 0)} = \frac{\sqrt{t}}{\sqrt{t - t_i}} \qquad (29)$$

By equating Eqs. (28) and (29), and substituting Eq. (20) for t with Eq. (26) for $u_e(x)$, the transient variable $t - t_i$ can be written as

$$t - t_i = \frac{2x}{\sqrt{2C_1 g\beta \bar{T}_w x}} \left[1 - \left(\frac{x_i}{x} \right)^a \right]^{2b} \qquad (30)$$

Substituting Eq. (30) with $a = \frac{3}{4}$ and $b = \frac{1}{3} + \frac{1}{8}$ into Eq. (16) results in

$$\eta_i = \frac{C\sqrt{\pi}}{2} \hat{G}r_x^{1/4} \frac{y}{x} \left[1 - \left(\frac{x_i}{x} \right)^{3/8} \right]^{-(1/3 + 1/8)} \qquad (31)$$

Since the forgoing derivation is developed for the arbitrary ith step change, Eq. (13) with η_i given by Eq. (31) represents the approximate temperature solution for the case with multistep changes. The complementary error function appearing in the solution can be evaluated by means of a numerical integration or by using a rational approximation[18] with a high degree of accuracy.

From the temperature solution, the local surface heat flux is obtained as

$$q_w = C \frac{k}{x} \bar{G}r_c^{1/4} \sum_{i=0}^{\kappa} \Delta T_{w_i} \left[1 - \left(\frac{x_i}{x} \right)^{0.8} \right]^{-(1.3 + 1/8)} \quad (32)$$

where C can be evaluated from Eq. (24).

Comparisons and Discussions

As with other approximate solutions developed in the past, the present solution needs to be validated by comparing the results with those that are obtained by employing "exact" solution techniques. The temperature distributions in the boundary layer for the case of an isothermal plate are compared in Fig. 2 with the similarity solutions of Ostrach[19] for various Prandtl numbers. Figures 3 and 4 show the comparisons of the dimensionless temperature field development in air $(Pr = 0.72)$ due to a single step change in surface temperature at selected downstream locations with step ratios $\theta_{w_1} = T_{w_1}/T_{w_0} = 0.503$ and 0, respectively. As can be seen from the figures, the present results agree well with those obtained by using the finite-difference methods[5] and experimental measurements.[4]

Natural convection results for a vertical plate with nonsimilar thermal conditions have been obtained by many investigators using different methods. The plate with a step jump in surface temperature has been examined extensively in the past. Although it has little practical importance, an examination of this case provides useful information to developers of new solution methods, especially when the method is approximate such as the one presented herein.

A dimensionless surface heat flux distribution for $x > x_1$ for the foregoing case can be written from Eq. (32) with $\kappa = 1$ as

$$q_{w_1}^* = \frac{q_{w_1}}{q_{w_0}} = \bar{\theta}_w^{1/4} \left[1 + (\theta_{w_1} - 1)(1 - \xi^{-0.8})^{-(1.3 + 1.8)} \right] \quad (33)$$

with

$$\bar{\theta}_w = 1 + (\theta_{w_1} - 1)(1 - \xi^{-1}) \quad (34)$$

where q_{w_0} is the local heat flux at the location of interest with the entire wall maintained at temperature T_{w_0}. The result obtained by using the above equation is plotted and compared with other data in Fig. 5. The values indicated by arrows are the asymptotic values at large ξ. They are obtained from

$$\lim_{\xi \to \infty} q_{w_1}^* = \theta_{w_1}^{5/4} \quad (35)$$

In practice, the laminar regime is not likely to be maintained at far downstream locations. Nevertheless, it is worthwhile stressing that the present solutions satisfy all the limiting conditions at large ξ, as it can be shown that the solution becomes similar based on T_{w_1} in the limit as $\xi \to \infty$. Figure 5 exhibits an excellent agreement of the present predictions especially
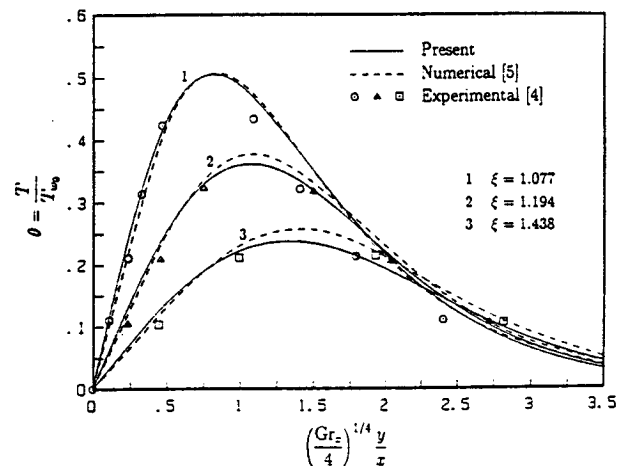

Fig. 4 Comparison of dimensionless temperature field development with a step change in wall temperature $(\theta_{w_1} = 0, Pr = 0.72)$.

Fig. 2 Comparison of dimensionless temperature distributions for various Prandtl numbers with a uniform wall temperature.
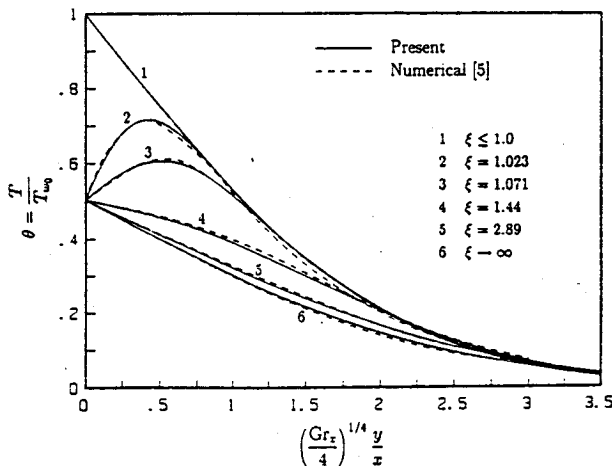
Fig. 3 Comparison of dimensionless temperature field development with a step change in wall temperature $(\theta_{w_1} = 0.503, Pr = 0.72)$.
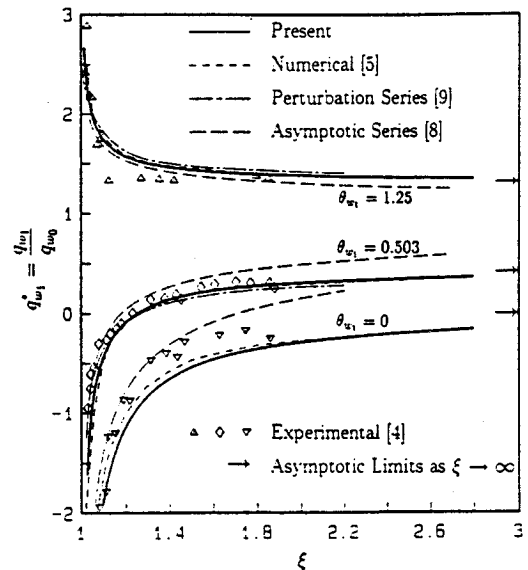

Fig. 5 Comparison of dimensionless surface heat flux with a step change in wall temperature $(Pr = 0.72)$.

with the numerical results of Hayday et al.[5] and the perturbation series solution of Kao.[9] Kao's solution is unavailable for $\theta_{w} = 0$ since his perturbation parameter becomes unbounded for this case. Considering the conditions at large ξ, improper deviations from the limiting values are observed in the asymptotic series solution of Kelleher.[8]

If a solution method is to become a valuable working tool for practical uses, not only should it be simple. but also it should be capable of making accurate predictions for the cases with a continuous surface temperature variation. In Figs. 6 and 7, $Nu_a/Gr_a^{1/4}$ predictions are plotted as a function of the dimensionless position x/x_a for surface temperature variations of the exponential and sinusoidal forms. respectively. The parameters T_a and x_a appearing in the figures represent arbitrary temperature and length scales. As was discussed previously, the continuous variation is discretized into a number of step changes and the surface heat flux at the midlocation of each element is computed. It is found that the solution at a given location does not depend strongly on the number of elements used. Nonetheless. a sufficiently large number of equally spaced elements is used in order to generate smooth plots. Also shown in the figures are the results of the local similarity,[14] local nonsimilarity[14] and numerical methods.[13,14] They are obtained by digitizing the plots presented in the referenced papers and converting the variables to the present forms. The numerical results of Kao et al.[14] completely overlap with those of Yang et al.[13] As can be seen from the figures, the agreement between the present solutions and the local nonsimilarity and numerical results are remarkably good. Except for the local similarity method, the results predicted by different methods are virtually indistinguishable particularly for the cases with the exponential variation.
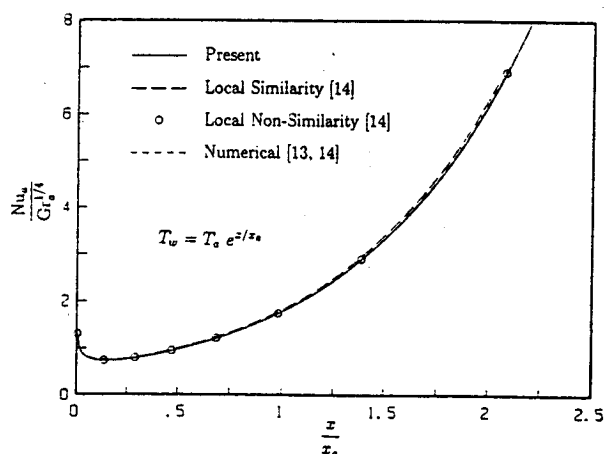

Fig. 6 Comparison of dimensionless surface heat flux with exponentially increasing wall temperature ($Pr = 0.7$).
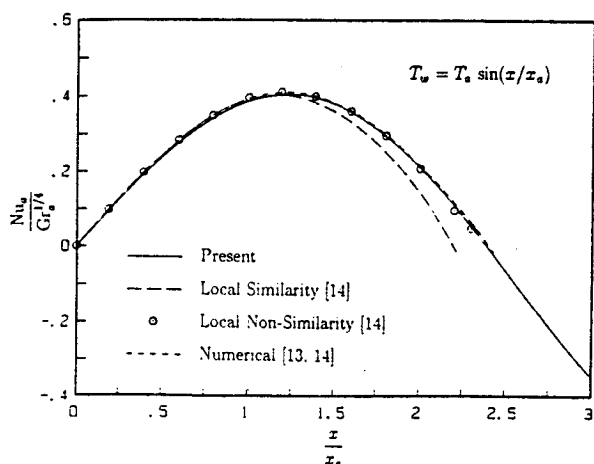

Fig. 7 Comparison of dimensionless surface heat flux with sinusoidal wall temperature variation ($Pr = 0.7$).

Summary and Concluding Remarks

An approximate. but simple and highly accurate model is developed for predicting the temperature distribution and the surface heat flux variation in laminar-free convection along a vertical plate with an arbitrarily specified wall temperature variation. The model is developed based on a transformation of the problem into a linearized form through the use of an effective velocity. The effective velocity characterizes the boundary-layer flow, and it is found to be analogous to the externally induced flow velocity. The accuracy and validity of the present solutions are demonstrated by comparing the results with the existing data obtained by using more rigorous techniques.

The study reveals the usefulness and potential capability of approximate techniques in producing adequate solutions for problems for which exact solutions are not available or are difficult to obtain. The method shows that it is possible to obtain linearized solutions for nonlinear natural convection problems with a high degree of accuracy. The methodology employed in developing the present temperature specified model is equally applicable in developing a model for surface heat-flux specified problems. Such solutions have been obtained by the current authors,[20] and they are also in excellent agreement with numerical[13,14] and local nonsimilarity solutions.[14]

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