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LINEARIZATION OF NATURAL CONVECTION FROM A VERTICAL PLATE WITH ARBITRARY HEAT-FLUX DISTRIBUTIONS

S. Lee and M. M. Yovanovich
Department of Mechanical Engineering
Microelectronics Heat Transfer Laboratory
University of Waterloo
Waterloo, Ontario, Canada

ABSTRACT

A simple model is developed to predict the heat transfer characteristics of a vertical plate with arbitrarily prescribed surface heat-flux variations under a buoyancy driven flow. The analysis is based on the linearized approximations to the boundary layer form of the conservation equations. Solutions for the surface temperature of the plate and fluid temperature distributions are obtained for air. The cases with a variety of different surface heat-flux distributions are examined, and the results are compared with those obtained by using methods that are capable of producing exact solutions. The agreement is excellent. The comparisons validate the use of the present model and demonstrate the capability of the current analysis to produce correct solutions.

NOMENCLATURE

b	exponent in equation (33)
C	coefficient given by equation (26)
C_1, C_2	coefficients
f_u	function defined by equation (15)
g	gravitational acceleration
Gr_a^*	modified Grashof number, $g\beta q_a x_a^4/k\nu^2$
Gr_x^*	modified Grashof number, $g\beta q_{w_0} x^4/k\nu^2$
$\overline{\mathrm{Gr}}_x^*$	modified Grashof number, $g\beta\overline{q}_w x^4/k\nu^2$
$\overline{h}$	average heat transfer coefficient
k	thermal conductivity
Nu_x	Nusselt number, $q_{w_0}x/T_w k$
Pr	Prandtl number, ν/α
q	local heat flux
q^*	dimensionless heat flux, q/q_{w_0}
$\overline{q}$	average heat flux defined by equation (32)
Q_w	heat dissipation defined by equation (30)
Re_x	Reynolds number, $u_e x/\nu$
t	variable defined by equation (9)
T	temperature excess over ambient fluid temperature
u	local velocity in the x-direction
u_e	effective flow velocity across the boundary layer
v	local velocity in the y-direction
x	coordinate along the plate
y	coordinate normal to the plate

Greek Symbols

α	thermal diffusivity of fluid
β	thermal expansion coefficient
δ	boundary layer thickness
Δq_{w_i}	heat-flux difference, $q_{w_i} - q_{w_{i-1}}$
ζ	dummy variable
η	similarity variable
θ	dimensionless temperature, equation (40)
ν	kinematic viscosity
ξ	dimensionless x-coordinate, x/x_1
ϕ	dependent variable, u or T
χ	delay factor

Subscripts

a	arbitrary references
i	parameters at i-th step
w	wall conditions
κ	parameters at κ-th step
0	parameters at leading section

INTRODUCTION

The natural convection heat transfer from a vertical plate has been a subject of numerous investigations in the past few decades. Plates with thermal conditions that allow similarity transformations have been examined by Ostrach (1953), Sparrow and Gregg (1956, 1958), and Jaluria and Gebhart (1977). They have considered steady-state, two-dimensional laminar boundary layer equations for a uniform wall temperature, a uniform surface heat flux, excess wall temperature variations of the power and exponential forms, and a line source on an adiabatic plate. Yang (1960) reported that there are no other types of boundary conditions which would make similarity solutions possible for steady natural convection heat transfer from a vertical plate.

Many more studies have followed so as to expand the number of available solutions for non-similar boundary conditions. For discontinuous temperature variations, Schetz and Eichhorn (1964) conducted an experiment with a Mach-Zehnder Interferometer. Hayday et al. (1967) and Sokovishin and Érman (1982) carried out numerical studies, and Smith (1970), Kelleher (1971), and Kao (1975) solved the same problem using series expansions. Jaluria (1982, 1985), employing finite difference methods, investigated cases with a number of uniform heat-flux sources on an adiabatic plate in air. Several approximating techniques are also available for surfaces with continuous thermal conditions. An integral method was employed by Sparrow (1955) and Scherberg (1964), and a Görtler-type series expansion was used by Kelleher and Yang (1972). Kao and his co-workers (1977) developed local similarity and local non-similarity methods for problems with arbitrarily specified surface thermal conditions and presented comparisons of the results with numerical solutions. The boundary layer equations with non-uniform surface thermal conditions were again solved numerically by Yang et al. (1982). They obtained Merk-type series solutions which resulted in values that are identical or very close to the numerical results of Kao et al. (1977) for a variety of surface conditions. More recently, Lee and Yovanovich (1989, 1991a), and Park and Tien (1990) also developed approximate methods for a vertical plate with changes in thermal conditions.

Except for numerical methods, such as the finite difference, finite volume and finite element methods, exact solution techniques for general cases of non-similar problems do not exist in the literature. For the cases with an arbitrary surface variation, the boundary layer equations may be modified through variable transformations but the resulting differential equations will remain partial with two independent space variables. Close-to-exact solutions to these partial differential equations can be found by means of direct numerical integrations, or approximate solutions may be obtained by employing either the local similarity or local non-similarity method (Kao et al., 1977). The local similarity method presumes that all the terms associated with one of the transformed variables are negligible and solves the resulting set of ordinary differential equations. The local non-similarity method is an improved version of the local similarity method. It expands the terms that are ignored in the local similarity method to an asymptotic type of series transforming the problem into successive sets of ordinary differential equations. In any case, the final solutions to the ordinary differential equations resulting from the similarity, local similarity or local non-similarity method must be obtained by means of numerical integrations.

As can be seen from comparisons with the numerical results presented by Kao et al. (1977), the local non-similarity method is capable of producing highly accurate solutions for most of the examples that were considered. It was found, however, that both local similarity and local non-similarity methods fail to predict the proper solutions for some cases.

Simple, general solutions are of a great value to those involved in practical applications where problems arise in such a way that the surface thermal condition is not known but is itself to be found. This interfacial condition established between the fluid and the solid has to be determined by solving the fluid side convection heat transfer problem and the solid side heat diffusion process in an iterative procedure. Such problems are called conjugate problems. They occur in many situations including the thermal modeling of printed circuit boards, and the iterative solution procedure necessitates that the convection model be capable of dealing with arbitrary surface thermal conditions. Numerous studies have been carried out examining cases where the wall temperature is prescribed and the heat flux is to be found. In many cases of practical interest, however, it is often necessary to determine the temperature of the wall given the surface heat flux.

In this paper, a new approximate method is developed based on the linearization of the governing differential equations. The linearization is performed by introducing an effective flow velocity which is subsequently determined by relating the total thermal energy dissipated into the fluid to the effective kinetic energy of the fluid flow. The effective flow velocity determined in this manner becomes analogous to an externally induced flow velocity thereby allowing the analysis to proceed in a way that is similar to forced convection analysis. The result is an extremely simple, explicit model that is capable of predicting laminar natural convection heat transfer from a flat, vertical plate dissipating energy with arbitrarily prescribed surface heat-flux variations. The model can be used for cases involving step discontinuities as well as a continuous variation in surface heat flux.

As with most approximate solutions, the validity of the present model can only be established by comparing the results with data obtained by using solution techniques that are capable of producing exact or close-to-exact solutions. The downstream wall temperature and heat transfer characteristics are compared with the data obtained by employing various solution methods for different surface heat-flux conditions. The present model, consisting of a summation of simple power terms, yields exceptionally accurate and stable solutions.

Problem Statement

The geometric configuration and coordinate system of the problem are depicted in Fig. 1a, where a vertical plate is shown with a continuous variation in surface heat flux. Also shown in the right side of the figure is the description of the hydrodynamic boundary layer whose thickness is greatly exaggerated in the y-direction. The plate is dissipating heat into an extensive, quiescent fluid which is assumed to be maintained at uniform temperature.

The conservation of mass, momentum and energy for two-dimensional, steady-state, laminar boundary layer flow yields a set of governing differential equations, expressed as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta T \quad (2)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \quad (3)$$

The boundary conditions associated with the above equations are

$$\begin{aligned} &\text{at } y = 0\,, \quad u = v = 0,\ -k\frac{\partial T}{\partial y} = q_w(x) \\ &\text{as } y \to \infty\,, \quad u \to 0,\ T \to 0 \\ &\text{at } x = 0\,, \quad u = T = 0 \end{aligned} \quad (4)$$

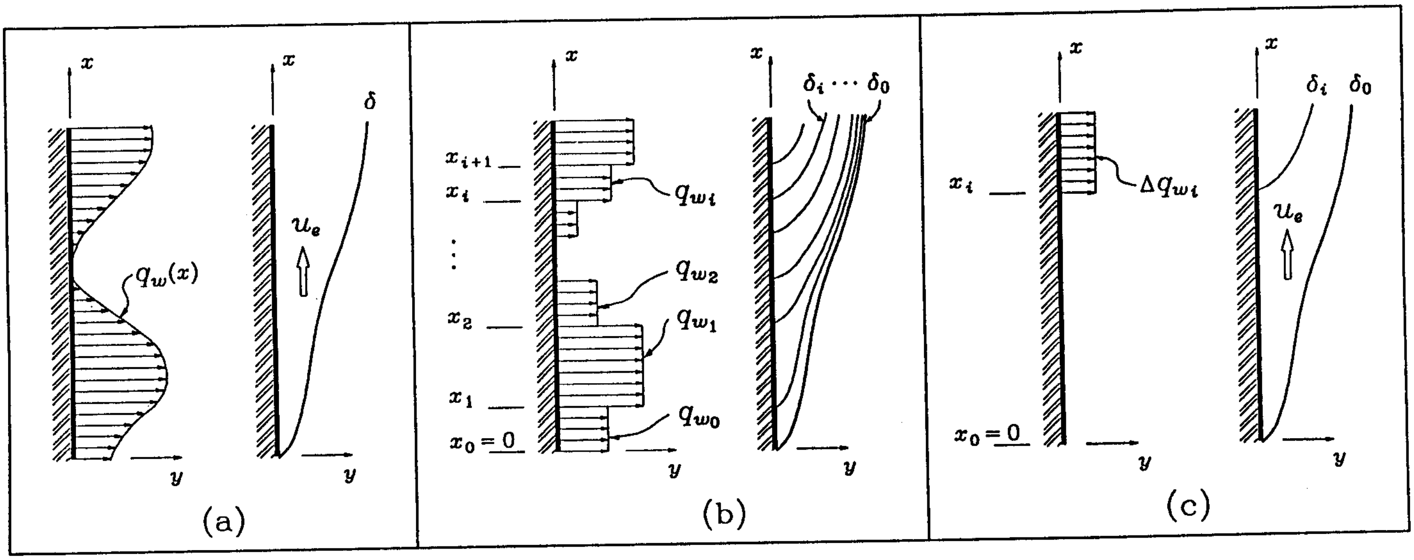

Figure 1 : Geometric configuration and coordinate system shown with schematic representations of surface heat flux variation and development of boundary layers where surface heat flux is prescribed by (a) continuous variation; (b) step changes; (c) an isolated step change in the wake of a buoyancy driven flow.

where T denotes the temperature excess over the ambient fluid temperature. The usual assumptions and approximations, such as those of constant fluid properties, except the density in the derivation of the buoyant term, and negligible viscous heating are made in deriving these equations.

Without a loss of generality, the surface heat-flux variation $q_w(x)$ is discretized into a number of step changes as depicted in Fig. 1b where only a few step changes are shown for brevity. The surface thermal condition given in equation (4) can be replaced by

$$-k\frac{\partial T}{\partial y}\bigg|_{y=0} = q_{w_i} \quad \text{for} \quad x_i \le x < x_{i+1} \ ; \ i = 0,1,2,... \tag{5}$$

where $x_i = 0$ when $i = 0$, and q_{w_i} is the average surface heat flux over the i-th section between $x = x_i$ and x_{i+1}. Obviously, a finer discretization would yield a closer representation of the continuous variation, and in such cases, q_{w_i} may be approximated simply by the heat flux at the mid-location of the section (i.e. $q_{w_i} = q_w$ at $x = (x_i + x_{i+1})/2$). It is to be noted from the figure that the sectional heat flux may be zero when the section is adiabatic. Furthermore, although it is not included in Fig. 1, the sectional heat flux may become negative when the fluid is heating the section, so long as the resulting vertical fluid velocity at any location within the boundary layer is maintained positive in the x-direction.

Also shown in Fig. 1b are the outer edges of the boundary layers developed by the onset of the step changes. The primary boundary layer δ, that is developed by the initial step jump of q_{w_0} at the leading edge of the plate, is denoted with a subscript 0. In addition to the primary boundary layer, the propagation of the effects induced by the introduction of a succeeding step change in surface heat flux into the fluid can be described as the growth of a boundary layer. This inner boundary layer grows within and blends into the outer layers, and the primary boundary layer, being the outer-most layer, encompasses all the inner layers at all times. Nonetheless, from an analysis point of view, it will be seen that each sub-layer can be isolated from the outer layers. The layers are denoted with the subscript which corresponds to that of the matching step change.

Figure 1c depicts an iso-flux plate with an adiabatic surface from the leading edge to $x = x_i$. Shown in the right side is the growth of the corresponding sub-layer within the wake of the primary boundary layer. The boundary layers depicted in Fig. 1 may be considered to be either momentum or thermal boundary layers since these two types of boundary layers would have similar patterns except for the differences in their relative thicknesses.

Linearization of Equations

An exact solution to the above problem for $x > x_1$ does not exist. An approximate solution is sought in which the non-linear convection terms appearing in the left side of the momentum and energy equations, equations (2) and (3), are linearized by introducing an effective flow velocity:

$$u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} \approx u_e(x)\frac{\partial \phi}{\partial x} \tag{6}$$

where ϕ represents the dependent variable, either u or T, and $u_e(x)$ is the effective flow velocity.

This approximation is applied to the differential equations and the effective flow velocity is defined to be uniform across the boundary layer. The consequence of this approximation is that the solutions to the linearized equations will not exactly describe the profiles of the dependent variables across the boundary layer. It is to be emphasized, however, that the correct forms of the solutions can still be obtained from the linearized equations. It requires finding the function $u_e(x)$ that properly describes the characteristics of the fluid flow in the x-direction. This can be accomplished by maintaining the momentum balance across the boundary layer.

By integrating the above equation from $y = 0$ to δ with $\phi = u$, the definition of $u_e(x)$ that is exact in the integral form of the momentum equation is obtained as

$$u_e(x) = \frac{\dfrac{d}{dx}\displaystyle\int_0^\delta u^2\,dy}{\dfrac{d}{dx}\displaystyle\int_0^\delta u\,dy} \tag{7}$$

Equation (1) was used in simplifying the numerator of the above expression. Alternative definition of $u_e(x)$, that is exact in the integral form of the energy equation, can also be obtained by integrating equation (6) with $\phi = T$. This would lead to an expression for $u_e(x)$ that involves the temperature field. But, as will become apparent, the resulting expression for $u_e(x)$ is functionally identical to the one obtained from equation (7). Since only the determination of the functional dependency of $u_e(x)$ is necessary at this stage of the analysis, equation (7) will be used in the following section in determining $u_e(x)$.

A transient variable t such that

$$dt = \frac{dx}{u_e(x)} \tag{8}$$

is introduced to transform the linearized convective operator given by the right side of equation (6). The above equation may be integrated to obtain the parameter t as

$$t = \int_0^x \frac{d\zeta}{u_e(\zeta)} \tag{9}$$

which can be viewed as the residence time of the flow from the leading edge of the plate to the downstream location x. With the use of the above newly defined variable, the original $x - y$ plane is transformed into the $t - y$ plane, and the momentum and energy equations become, respectively,

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} + g\beta T \tag{10}$$

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{11}$$

Recall that the continuity equation, equation (1), was absorbed in defining $u_e(x)$ and is, therefore, no longer necessary in the $t - y$ plane. The thermal boundary condition given by equation (5) becomes

$$-k\left.\frac{\partial T}{\partial y}\right|_{y=0} = q_{w_i} \quad \text{for} \quad t_i \leq t < t_{i+1} \; ; \; i = 0, 1, 2, ... \tag{12}$$

where t_i corresponds to $x = x_i$.

The most notable features that are different in the above transformed equations from the steady-state equations are the linearity of the dependent variables u and T, and the decoupling of the temperature field from the velocity field in the $t - y$ plane. The solutions to the above equations will not be identical to those of the original boundary layer equations. It is worthwhile, however, for the establishment of the present analysis to remind that the above equations will result in the correct forms of the exact solutions once the proper expression of $u_e(x)$ is found according to the definition given by equation (7).

ANALYSIS

The solutions to the linearized problem defined by equations (10) to (12) with the rest of the boundary conditions specified in equation (4) can be found by means of similarity transformations and the method of superposition (Lee, 1988). They are expressed in terms of the complementary error functions, for the κ-th step change for $x_\kappa \leq x < x_{\kappa+1}$, as

$$T = \frac{2\sqrt{\alpha}}{k}\sum_{i=0}^{\kappa}(q_{w_i} - q_{w_{i-1}})(t - t_i)^{1/2} \text{ ierfc } \eta_i \tag{13}$$

$$u = \frac{4\sqrt{\alpha} g\beta}{k}\sum_{i=0}^{\kappa}(q_{w_i} - q_{w_{i-1}})(t - t_i)^{3/2} f_u(\eta_i) \tag{14}$$

where

$$f_u(\eta) = \begin{cases} \eta \, \text{i}^2\text{erfc } \eta & \text{for } \Pr = 1 \\ \dfrac{2}{1 - \Pr}\left(\text{i}^3\text{erfc } \eta - \text{i}^3\text{erfc } \dfrac{\eta}{\sqrt{\Pr}}\right) & \text{for } \Pr \neq 1 \end{cases} \tag{15}$$

$$\eta_i = \frac{y}{2\sqrt{\alpha(t - t_i)}} \tag{16}$$

Here, and throughout the paper, $q_{w_{i-1}}$ vanishes when $i = 0$, and $t_0 = 0$.

The local surface temperature can be obtained by substituting $y = 0$ into equation (13). It follows that

$$T_w = \frac{2\sqrt{\alpha}}{k\sqrt{\pi}}\sum_{i=0}^{\kappa}(q_{w_i} - q_{w_{i-1}})(t - t_i)^{1/2} \tag{17}$$

where ierfc $0 = 1/\sqrt{\pi}$ was used.

The above solutions are exact solutions of the transformed equations which are an approximate representation of the original steady-state problem. The only unknown in the above solutions is the transient variable $t - t_i$. Therefore, determination of $t - t_i$ in terms of the known steady-state variables would render these solutions as approximate solutions to the original problem in the $x - y$ plane.

Consider the case of uniform surface heat flux for $0 \leq x < x_1$. Although exact similarity solutions are available (Sparrow and Gregg, 1956), the case will be examined in the following. The formulations derived for the uniform surface condition will provide the basis upon which the derivation can be extended to the general situation involving multi-step changes.

Substituting the first term of equation (14) into equation (7) and evaluating the integrals with $\delta \to \infty$ leads to

$$u_e(x) = C_1(\Pr)\,\frac{\sqrt{\alpha} g\beta\, q_{w_0}}{k}\, t^{3/2} \tag{18}$$

where $C_1(\Pr)$ is a function of the Prandtl number. Determination of this function is not necessary, and it is sufficient to note that this coefficient is a constant for a given fluid.

The above equation is differentiated after taking a 2/3 power of both sides. The resulting expression, after substituting equation (8) for dt and separating the variables, is

$$u_e\, du_e^{2/3} = \left(\frac{C_1\sqrt{\alpha} g\beta q_{w_0}}{k}\right)^{2/3} dx \tag{19}$$

Upon integration, this yields

$$u_e(x) = \left(\frac{C_1\sqrt{\alpha} g\beta q_{w_0}}{k}\right)^{2/5}\left(\frac{5x}{2}\right)^{3/5} \tag{20}$$

By substituting equation (20) into equation (9) for $u_e(x)$ and evaluating the integral, it can be shown that

$$t = \frac{5}{2}\frac{x}{u_e(x)} \tag{21}$$

Now, with $u_e(x)$ from equation (20), the transient variable t is completely determined in terms of the steady-state variables except the value of C_1.

The surface temperature variation for the iso-flux case is rewritten from equation (17) with $\kappa = 0$:

$$T_w = \frac{2\sqrt{\alpha t}\, q_{w_0}}{\sqrt{\pi} k} \quad (22)$$

By substituting equation (21) for t into the above equation, the local Nusselt number can be expressed as

$$\mathrm{Nu}_x = C_2(\mathrm{Pr})\ \mathrm{Re}_x^{1/2} \quad (23)$$

where $C_2(\mathrm{Pr})$ is another function of the Prandtl number, and Re_x denotes the Reynolds number defined based on $u_e(x)$ as

$$\mathrm{Re}_x = \frac{u_e x}{\nu} \quad (24)$$

Note that the form of the above heat transfer expression is identical to that of the forced convection result. It reveals that $u_e(x)$ is analogous to the externally induced free stream velocity in the forced convection study. By substituting equation (20) for u_e, equation (23) becomes

$$\mathrm{Nu}_x = C(\mathrm{Pr})\mathrm{Gr}_x^{*1/5} \quad (25)$$

where $C(\mathrm{Pr})$ consists of $C_1(\mathrm{Pr})$ and $C_2(\mathrm{Pr})$. This is again identical to the form of known results for laminar natural convection heat transfer from an iso-flux surface. The Prandtl number function $C(\mathrm{Pr})$ can be determined by conserving the momentum and energy over the boundary layer (Lee, 1988) or simply by comparing the above heat transfer result with known data. Since the prime objective of the present investigation is to seek a simple solution that requires the least computational effort, the latter will be adopted to determine $C(\mathrm{Pr})$. For example, by comparing equation (25) with the correlation equation proposed by Fujii and Fujii (1976), it can be found that

$$C(\mathrm{Pr}) = \left(\frac{\mathrm{Pr}^2}{4 + 9\sqrt{\mathrm{Pr}} + 10\ \mathrm{Pr}}\right)^{1/5} \quad (26)$$

With the use of equations (20), (21) and (26), the temperature solution for iso-flux cases can now be expressed completely in terms of the known variables and coefficient. There follows

$$T = \frac{\sqrt{\pi}\, x}{C\, k\, \mathrm{Gr}_x^{*1/5}} q_{w_0}\ \mathrm{ierfc}\, \eta_0 \quad (27)$$

with

$$\eta_0 = \frac{C}{\sqrt{\pi}} \mathrm{Gr}_x^{*1/5} \frac{y}{x} \quad (28)$$

The analysis so far has demonstrated that the correct heat transfer relationship can be obtained for iso-flux cases from the linearized differential equations by properly determining the effective flow velocity. An extension of the above derivations for problems involving multiple step changes in surface heat flux is outlined as follows.

Recall that, in linearizing the governing differential equations, the effective flow velocity $u_e(x)$ was introduced as an input parameter to the equations and, therefore, it needs to be determined based on information external to the differential equations. In the above example with a uniform heat-flux surface, this was accomplished by equating the integral forms of the equations. It resulted in equation (19) which was subsequently integrated to obtain equation (20) for $u_e(x)$.

In terms of the total heat input Q_w into the fluid, equation (20) can be rewritten as

$$u_e^2 \propto Q_w^{4/5} x^{2/5} \quad (29)$$

where

$$Q_w = \int_0^x q_w(\zeta)\, d\zeta \quad (30)$$

Equation (29) reveals that the effective kinetic energy of the fluid flow at an arbitrary location x is related to the total heat dissipation into the fluid from the leading edge of the plate to the location of interest.

An assumption is made such that the relationship indicated by equation (29) between the effective kinetic energy of the flow and the cumulative heat input into the fluid is not affected by the existence of a variable heat flux at the surface. Therefore, the effective flow velocity $u_e(x)$ for the cases with an arbitrarily varying $q_w(x)$ may be written in a form that is similar to equation (20):

$$u_e(x) = \left(\frac{C_1\sqrt{\alpha} g \beta \overline{q}_w}{k}\right)^{2/5} \left(\frac{5x}{2}\right)^{3/5} \quad (31)$$

where $\overline{q}_w$ is the average surface heat flux defined as

$$\overline{q}_w(x) = \frac{Q_w}{x} \quad (32)$$

As was previously indicated in the case with an iso-flux plate, the velocity defined by equation (31) is analogous to the induced free stream velocity in the forced convection analysis. In the following, the effect that each isolated step jump in surface heat flux has on the overall temperature solution and the heat transfer characteristics of the plate will be determined.

Consider the flow over a vertical plate which experiences a single step jump in surface heat flux $\Delta q_{w_i} = q_{w_i} - q_{w_{i-1}}$ at some downstream location $x = x_i$. This is a classical problem that appears in the study of forced convection heat transfer from a flat plate which has an unheated starting length in the wake of a hydrodynamic boundary layer as shown in Fig. 1c. The primary momentum boundary layer growth denoted by δ_0 begins at the leading edge whereas the thermal boundary layer development begins at $x = x_i$. Through use of an integral boundary layer solution (Sparrow and Lin, 1965) it can be shown that the delay factor χ which accounts for the unheated starting-length effect can be obtained through the ratio of the thermal-to-hydrodynamic boundary layer thicknesses. It is found that

$$\chi = \left(1 - \frac{x_i}{x}\right)^b \quad (33)$$

with $b = 1/3$ for the externally induced laminar flow of a high Prandtl number fluid.

The above delay factor is obtained by regarding the entire flow over the heated section as externally induced. Therefore, the effect that the buoyant force due to the introduction of the step heating over the surface has on the fluid flow within the i-th boundary layer is not accounted for in the derivation of the delay factor. Since the value of b depends on the type of flow it is reasonable to assume that b should be modified in order to incorporate this buoyancy effect into the delay factor. By considering the limiting values of the Prandtl number and the definition of the parameter t given by equation (9), it is found that the value of b must lie in the range of $1/3 < b < 1/2$. An examination of heat transfer results confirms that, for air (Pr $\approx$ 0.7), $b = 1/3 + 1/10$ results in the best agreement with existing data obtained by various techniques for different surface heat-flux distributions.

Recall that the delay factor is proportional to the ratio of the thermal-to-hydrodynamic boundary layer thicknesses. Also, by noting from equation (16) that the growth of the boundary layer thickness varies with the square root of the transient variable, it can be written that

$$\chi = \frac{\sqrt{t - t_i}}{\sqrt{t}} \tag{34}$$

Upon equating equations (33) and (34), one finds

$$t - t_i = t\left(1 - \frac{x_i}{x}\right)^{2b} \tag{35}$$

Here, the parameter t is related to the growth of the primary boundary layer. By substituting equation (21) for t with equation (31) for $u_e(x)$, the above equation becomes

$$t - t_i = \left(\frac{5}{2}\frac{kx}{C_1\sqrt{\alpha g \beta \,\overline{q}_w}}\right)^{2/5}\left(1 - \frac{x_i}{x}\right)^{2b} \tag{36}$$

with $b = 1/3 + 1/10$ as discussed previously.

Using equation (36) for $t - t_i$, the parameters associated with the temperature solution given by equation (13) are completely determined. The steady-state temperature solution becomes

$$T = \frac{\sqrt{\pi}\,x}{C\,k\,\overline{\mathrm{Gr}}_x^{*1/5}} \sum_{i=0}^{\kappa}(q_{w_i} - q_{w_{i-1}})\left(1 - \frac{x_i}{x}\right)^{1/3+1/10} \mathrm{ierfc}\,\eta_i \tag{37}$$

where

$$\eta_i = \frac{C}{\sqrt{\pi}}\overline{\mathrm{Gr}}_x^{*1/5}\frac{y}{x}\left(1 - \frac{x_i}{x}\right)^{-(1/3+1/10)} \tag{38}$$

and $\overline{\mathrm{Gr}}_x^*$ is the modified Grashof number based on the average surface heat flux $\overline{q}_w$. The complementary error function appearing in the solution can be evaluated by means of a numerical integration or by using a rational approximation (Abramowitz and Stegun, 1972) with a high degree of accuracy.

The local wall temperature over the κ-th step for $x_\kappa \le x < x_{\kappa+1}$ becomes

$$T_w(x) = \frac{x}{C\,k\,\overline{\mathrm{Gr}}_x^{*1/5}} \sum_{i=0}^{\kappa}(q_{w_i} - q_{w_{i-1}})\left(1 - \frac{x_i}{x}\right)^{1/3+1/10} \tag{39}$$

where C can be evaluated from equation (26). The above solutions include the iso-flux plate solutions for which $\kappa = 0$ and $\overline{\mathrm{Gr}}_x^* = \mathrm{Gr}_x^*$.

Comparisons and Discussions

Many simplifying approximations, such as the linearization of the governing differential equations and the assumption of a non-variant proportionality relationship between the effective kinetic energy of the flow and the total heat dissipation, were introduced into the development of the model. Since the present solutions are based on a novel, approximate method, a major portion of this section will be devoted to comparisons with other data obtained by using various techniques. It is to be recalled that the intent of the study was to develop the simplest possible solutions that are sufficiently accurate for practical uses in predicting temperature distributions and surface heat transfer characteristics. As with other approximate solutions developed in the past, the use of the present solutions can only be validated by comparing the results with those that are obtained by employing more exact methods.

To examine the solutions for step changes in surface heat flux, a case in air (Pr = 0.7) with a number of strip heat sources mounted flush with the surface of an adiabatic plate is considered. A three-dimensional plot for the dimensionless temperature distribution in the fluid is presented in Fig. 2. All five sources shown in this figure have a uniform surface heat flux of equal strength and size, with non-source spaces equal to the source size; q_{w_i} is positive for $i = 0$ or even, and is 0 for i = odd; and all step sizes $(x_{i+1} - x_i)$ for $i = 0$ to 9 are identical. The dimensionless temperature θ appearing in the figure is defined as

$$\theta = \frac{T}{q_{w_0} x_1 / k \mathrm{Gr}_{x_1}^*} \tag{40}$$

and $\xi = x/x_1$.

Since the boundary layer approximations assume no thermal energy transport by diffusion along the x-direction, the temperature distributions shown in Fig. 2 would be less valid in the immediate vicinity of a step change where large temperature gradients are observed. Jaluria, using finite difference methods, solved the boundary layer equations (Jaluria, 1982) as well as the full elliptic equations (Jaluria, 1985) and presented the effects that the boundary layer approximations have on both surface temperature and maximum velocity variations for different modified Grashof numbers. The studies revealed, as expected, that the local peak temperatures at the locations of a step change dampens as the value of the modified Grashof number decreases.

From equation (39), the dimensionless surface temperature can be written as

$$\theta_w = \frac{\xi^{1/5}}{C\,\overline{q}_w^{*\,1/5}} \sum_{i=0}^{\kappa}(q_{w_i}^* - q_{w_{i-1}}^*)\left(1 - \frac{\xi_i}{\xi}\right)^{1/3+1/10} \tag{41}$$

where $q_w^* = q_w/q_{w_0}$ and $\overline{q}_w^* = \overline{q}_w/q_{w_0}$. This equation is evaluated for the previous case with five heat sources. As can be seen from Fig. 3, the results are in excellent agreement with the numerical data of STAN7. STAN7 is a finite difference code developed for solving the boundary layer momentum and energy equations for laminar and turbulent flows over or inside a body of revolution, and it is an abridged version of program STAN5 (Crawford and Kays, 1976).

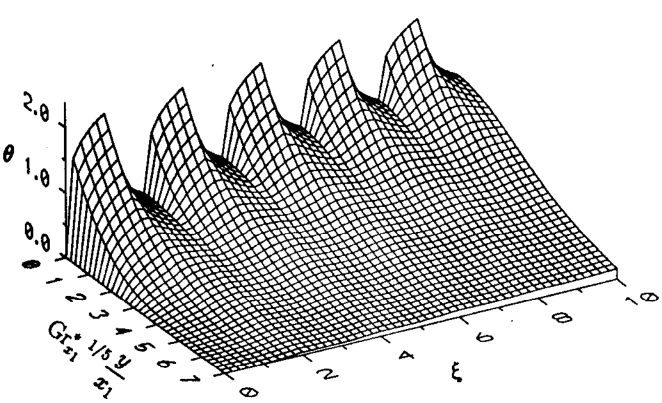

Figure 2 : Dimensionless temperature distribution due to alternating positive uniform and zero surface heat fluxes of equal size for Pr = 0.7.

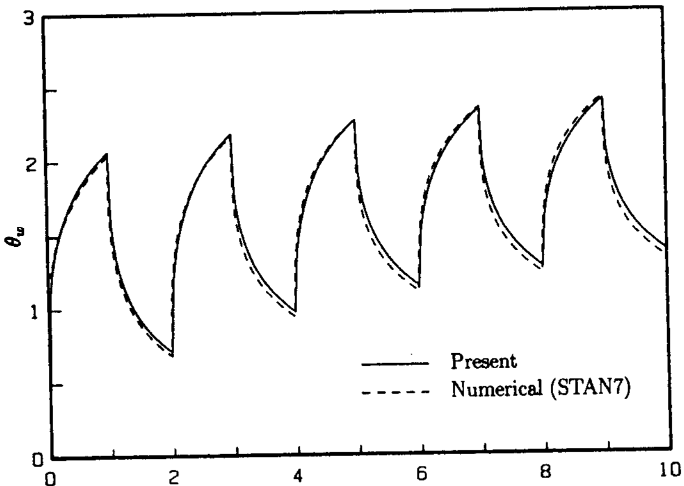

Figure 3 : Comparison of dimensionless wall temperature variation due to alternating positive uniform and zero surface heat fluxes of equal size for Pr = 0.7.

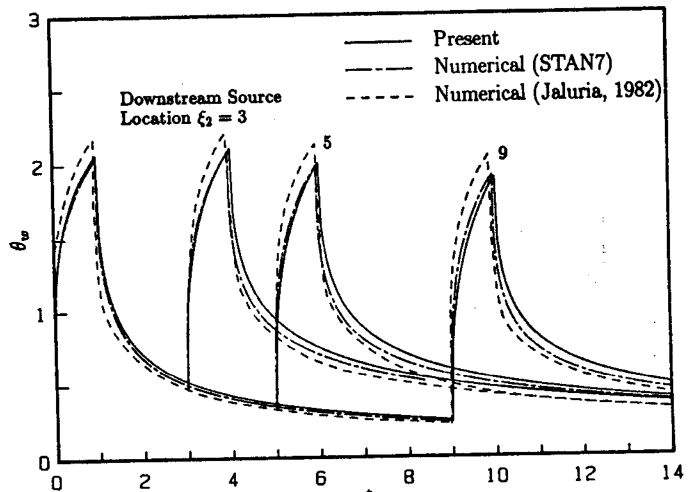

Figure 4 : Comparison of dimensionless wall temperature variations due to two identical thermal sources of uniform heat flux on an adiabatic plate with different source spacings for Pr = 0.7.

Figure 4 presents a similar situation with two identical iso-flux heat sources for Pr = 0.7 showing the effect of the source spacings on the wall temperature variations. One source is at the leading edge of the plate and the other is located downstream. The center-to-center source distance is equal to ξ_2 times the source width. Three different source spacings are examined. Comparisons with the numerical data of Jaluria (1982) and STAN7 show good agreement, particularly in the trend of the local peak temperatures. Small deviations are observed in the numerical data of Jaluria who solved the boundary layer equations using finite difference methods. Jaluria's numerical prediction overestimates the peak temperature over the first source by approximately 5% when it is compared to the present solution. The present solution is identical, by choice, to the correlation equation of Fujii and Fujii (1976) for $\xi \leq 1$.

An examination of Fig. 4 also reveals the effects of the non-linearity of natural convection. Although the two sources are identical, dissipating the same amount of heat into the fluid over the same width, the peak temperature over the downstream source can become lower than that of the upstream source. The wall temperature due solely to the first source decreases asymptotically to the ambient fluid temperature as ξ increases, and the fluid temperature excess within the boundary layer remains positive. The positive temperature excess results in a positive buoyant force which, in turn, results in a perpetual increase in the overall downstream flow velocity. Despite the fact that the fluid flowing towards the downstream source is at a temperature higher than the ambient temperature, its increased flow velocity enhances the heat transfer from the second source. The result is that the downstream source temperature becomes lower as the source spacing increases.

For problems with two heat sources of uniform but different fluxes on an adiabatic plate ($q_{w_0} \neq q_{w_2}$ and $q_{w_1} = 0$), a ratio of the average heat transfer coefficient of the downstream to upstream sources can be defined as

$$\frac{\overline{h}_2}{\overline{h}_0} = q^*_{w_2} \frac{\int_{\xi_2}^{\xi_2+1} \frac{d\xi}{\theta_w}}{\int_0^1 \frac{d\xi}{\theta_w}} \tag{42}$$

where the integral in the denominator can be integrated to yield $5\mathcal{C}/4$, and equation (41) can be used for θ_w in the numerical integration of the numerator. This ratio is evaluated as a function of ξ_2 for $q^*_{w_2} = 0.1$, 0.2, 1 and 2, and Pr = 0.7. The results are plotted in Fig. 5, and compared again with the numerical data of Jaluria (1982) and STAN7. Considering that Jaluria (1982) also solved the boundary layer equations, the agreement between the three sets of data is poor. Better agreement of the present model is obtained, however, when it was compared with STAN7, especially for small flux ratios.

When the surface heat flux of the downstream source is equal to that of the upstream source ($q^*_{w_2} = 1$) and the space between the two sources approaches zero ($\xi_1 \to 1$), the exact solution to the above heat transfer ratio can be found. This case is nothing more than an iso-flux case with twice the source width. The uniform flux solution can be used over both sources, and the exact heat transfer ratio can be calculated as the constant value 0.741. As can be seen from Fig. 5, the present model and STAN7 correctly predict the value for this situation.

The rest of this section examines the cases with a continuous surface heat-flux variation. To generalize the solutions to account for any arbitrary variation of $q_w(x)$, which may include a sectionally continuous and varying function, the summation in the solutions may be replaced by a Stieltjes integral. For example, the surface temperature solution given by equation (39) can be generalized as

$$T_w(x) = \frac{x}{\mathcal{C}\, k\, \overline{\mathrm{Gr}}_x^{*1/5}}\, I_s \tag{43}$$

where

$$I_s = \int_{\zeta=0}^{x} \chi(\zeta,\, x)\; dq_w(\zeta) \tag{44}$$

with

$$\chi(\zeta,\, x) = \left(1 - \frac{\zeta}{x}\right)^{1/3+1/10} \tag{45}$$

I_s defined by equation (44) is the Stieltjes integral. It includes the ordinary Riemann integral plus contributions which occur whenever $q_w(x)$ has a discontinuity:

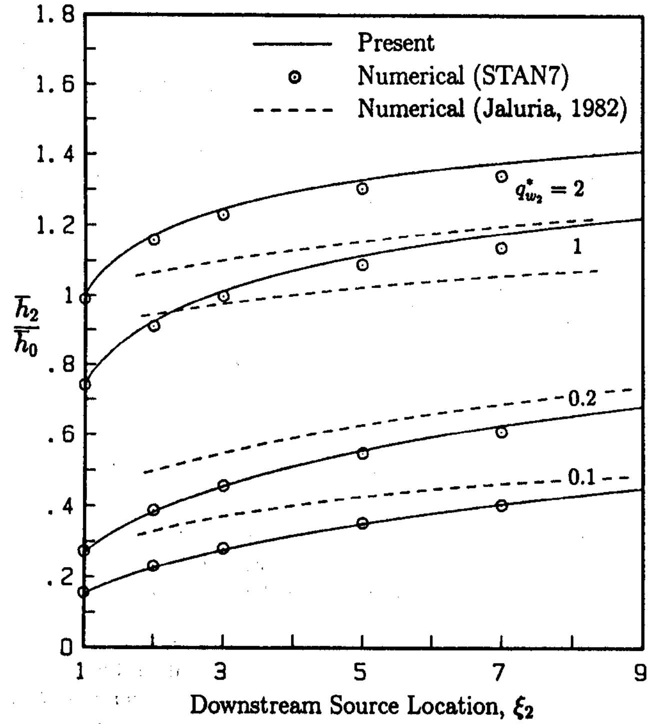

Figure 5 : Comparison of heat transfer coefficient with two thermal sources of uniform heat flux on an adiabatic plate showing the effects of flux ratios and source spacing for $\mathrm{Pr} = 0.7$.

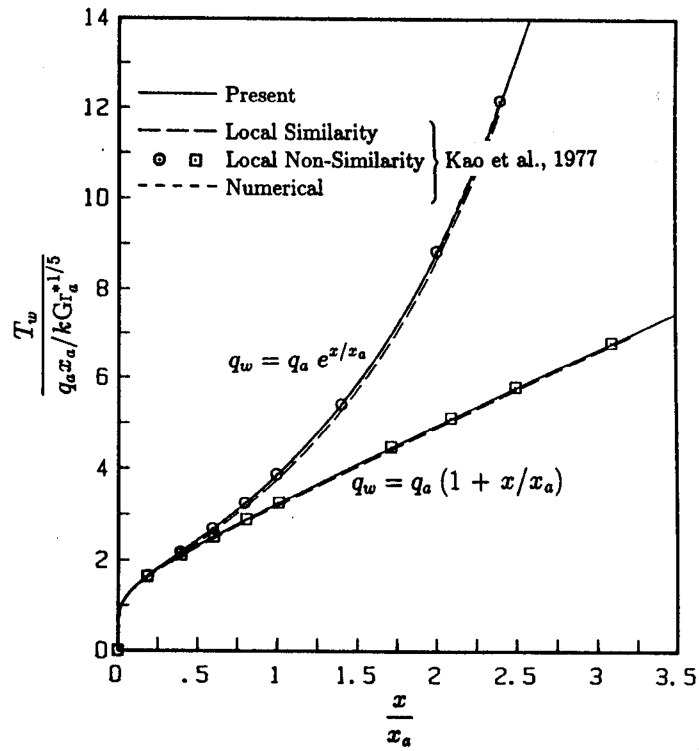

Figure 6 : Comparison of dimensionless wall temperature variations with exponentially and linearly increasing surface heat fluxes for $\mathrm{Pr} = 0.7$.

$$I_s = \int_0^x \chi(\zeta, x)\frac{dq_w(\zeta)}{d\zeta}\,d\zeta + \sum \chi(\zeta_i, x)[q_w(\zeta_i^+) - q_w(\zeta_i^-)] \quad (46)$$

Here, the summation represents all jumps in q_w at $x = \zeta_i$.

While the above integral form of the solutions is general, it is almost always the case that numerical computations must be carried out in evaluating the integrals. During the course of the numerical evaluation, the integral will be replaced by a summation reducing the solution back to the summation form given by equation (39). Therefore, it is apparent that equation (39) can be used effectively also for the cases with an arbitrarily varying $q_w(x)$.

Surface heat fluxes in the form of a linearly increasing or decreasing function and exponentially increasing surface heat flux are selected as test cases. They are given by

$$\frac{q_w(x)}{q_a} = 1 \pm \frac{x}{x_a} \quad (47)$$

and

$$\frac{q_w(x)}{q_a} = e^{x/x_a} \quad (48)$$

where q_a and x_a are arbitrary heat-flux and length scales, respectively.

A prescribed surface heat-flux variation is discretized and replaced by step changes in uniform surface heat flux. In this study, an equal spacing is used in the discretization and the heat-flux value at the mid-location of each element is used to represent the uniform heat flux over the element. Experimental computations were carried out to observe the effect of discretization, and it was found that the computed values at fixed locations are insensitive to the element size. Nevertheless, a sufficiently large number of elements is used in order to generate smooth plots for the surface temperature distributions.

Figure 6 compares the results of the dimensionless wall temperature for the linearly and exponentially increasing surface heat flux, and Fig. 7 compares the linearly decreasing surface heat flux with those obtained by using the local similarity, local non-similarity and a difference-differential numerical methods (Kao et

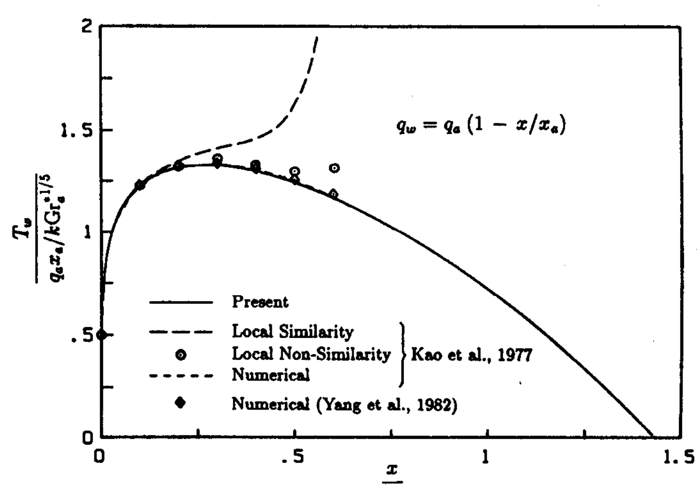

Figure 7 : Comparison of dimensionless wall temperature variations with linearly decreasing surface heat flux for $\mathrm{Pr} = 0.7$.

al., 1977). Also included in Fig. 7 is the numerical data of Yang et al. (1982). The data of Kao et al. and those of Yang et al. are obtained by digitizing the plots presented in their papers and converting the variables to the present forms. Although Yang et al. (1982) also reported numerical solutions for the cases presented in Fig. 6, the results are virtually identical to the numerical data of Kao et al. (1977) and are not included in Fig. 6 for clarity. Figure 6 exhibits an excellent agreement amongst all methods with small differences between the local similarity solutions and others. As can be seen from Fig. 6, the present predictions are indistinguishable from the results of numerical and local non-similarity methods. Figure 7 reveals the possibility of diverging solutions that can result from both local similarity and local non-similarity methods, whereas the present solution continues to maintain the close agreement with the numerical solutions of Kao et al. (1977) and Yang et al. (1982), which are reported up to $x/x_a = 0.643$.

Summary and Conclusions

An approximate method of determining the temperature distributions in laminar free convection along a vertical plate with an arbitrarily specified surface heat flux has been described. This new technique is based on a transformation of the problem into a linearized form by introducing an effective flow velocity which is determined by relating the effective kinetic energy of the fluid flow to the total heat dissipation into the fluid. The effective flow velocity determined in this manner is found to be analogous to an externally induced flow velocity. Therefore, by using the integral solution technique developed for forced convection problems, the delay factor that accounts for the effects of an incremental step change in surface heat flux on the temperature solution is obtained. The resulting model is a simple, explicit expression which only requires a summation of a finite number of algebraic terms for computing surface temperature distributions as a function of downstream location. The present solutions are in excellent agreement with other data obtained for air and different surface thermal conditions.

The same methodology described herein for heat-flux specified problems can also be applied for temperature specified problems. Such a study has been carried out by the current authors (Lee and Yovanovich, 1991b) and an equally simple and accurate expression has been obtained for predicting heat-flux distributions of a vertical plate with an arbitrary temperature variation prescribed along the surface.

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References

Abramowitz, M., and Stegun, I. A., 1972, *Handbook of Mathematical Functions,* Dover Publications, New York.

Crawford, M. E., and Kays, W. M., 1976, "STAN5 - A Program for Numerical Computation of Two-Dimensional Internal and External Boundary Layer Flows," NASA CR-2742.

Fujii, T., and Fujii, M., 1976, "The Dependence of Local Nusselt Number on Prandtl Number In the Case of Free Convection Along a Vertical Surface With Uniform Heat Flux," *Int. J. Heat Mass Transfer,* Vol. 19, pp. 121-122.

Hayday, A. A., Bowlus, D. A., and McGraw, R. A., 1967, "Free Convection From a Vertical Flat Plate With Step Discontinuities in Surface Temperature," ASME *Journal of Heat Transfer,* Vol. 89, No. 3, pp. 244-250.

Jaluria, Y., 1982, "Buoyancy-Induced Flow Due to Isolated Thermal Sources on a Vertical Surface," ASME *Journal of Heat Transfer,* Vol. 104, pp. 223-227.

Jaluria, Y., 1985, "Interaction of Natural Convection Wakes Arising From Thermal Sources on a Vertical Surface," ASME *Journal of Heat Transfer,* Vol. 107, pp. 883-892.

Jaluria, Y., and Gebhart, B., 1977, "Buoyancy-Induced Flow Arising From a Line Thermal Source on an Adiabatic Vertical Surface," *Int. J. Heat and Mass Transfer,* Vol. 20, pp. 153-157.

Kelleher, M., 1971, "Free Convection From a Vertical Plate With Discontinuous Wall Temperature," ASME *Journal of Heat Transfer,* Vol. 93, pp. 349-356.

Kelleher, M., and Yang, K. T., 1972, "A Görtler-Type Series for Laminar Free Convection along a Non-Isothermal Vertical Plate," *The Quarterly J. of Mechanics and Applied Mathematics,* Vol. 25, pp. 445-457.

Kao, T. T., 1975, "Laminar Free Convective Heat Transfer Response Along a Vertical Flat Plate With Step Jump in Surface Temperature," *Lett. Heat Mass Transfer,* Vol. 2, No. 5, pp. 419-428.

Kao, T. T., Domoto, G.A., and Elrod, H. G. Jr., 1977, "Free Convective Along a Nonisothermal Vertical Flat Plate," ASME *Journal of Heat Transfer,* Vol. 99, pp. 72-78.

Lee, S., 1988, "Laminar Natural Convection From a Vertical Plate With Variations in Thermal Boundary Conditions," PhD Thesis, Department of Mechanical Engineering, University of Waterloo, Ontario.

Lee, S., and Yovanovich, M. M., 1989, "Natural Convection From a Vertical Plate With Step Changes in Surface Heat-Flux," ASME HTD-Vol. 107, *Heat Transfer in Convective Flows,* pp. 239–247, Proceedings of the 26th National Heat Transfer Conference.

Lee, S., and Yovanovich, M. M., 1991a, "Laminar Natural Convection From a Vertical Plate With a Step Change In Wall Temperature," ASME *Journal of Heat Transfer,* Vol. 113, pp. 501–504.

Lee, S., and Yovanovich, M. M., 1991b, "Linearization Method For Buoyancy Induced Flow Over a Non-Isothermal Vertical Plate," AIAA paper No. 91-1308, AIAA 26th Thermophysics Conference.

Ostrach, S., 1953, "An Analysis of Laminar Free-Convection Flow and Heat Transfer About a Flat Plate Parallel to the Direction of the Generating Body Force," NACA Report No. 1111.

Scherberg, M. G., 1964, "Natural Convection From Wall Sections of Arbitrary Temperature Distribution by an Integral Method," *Int. J. Heat and Mass Transfer,* Vol. 7, pp. 501-516.

Park, S. H., and Tien, C. L., 1990, "An Approximate Analysis for Convective Heat Transfer on Thermally Nonuniform Surfaces," ASME *Journal of Heat Transfer,* Vol. 112, pp. 952–958.

Schetz, J. A., and Eichhorn, R., 1964, "Natural Convection With Discontinuous Wall-Temperature Variations" *J. Fluid Mech.,* Vol. 18, part 2, pp. 167-176.

Smith, R. K., 1970, "The Laminar Free-Convection Boundary Layer on a Vertical Heated Plate in the Neighbourhood of a Discontinuity in Plate Temperature," *J. Austral. Math. Soc.,* Vol. 11, No. 2, pp. 149-168.

Sokovishin, Y. A., and Érman, L. A., 1982, "Free-Convection Heat Transfer on a Vertical Surface With a Temperature Discontinuity," *J. Eng. Phy.,* Vol. 43, No. 2, pp. 858–861.

Sparrow, E. M., 1955, "Laminar Free Convection on a Vertical Plate With Prescribed Non-Uniform Wall Heat Flux or Prescribed Non-Uniform Wall Temperature," NACA TN-3508.

Sparrow, E. M., and Gregg, J. L., 1956, "Laminar Free Convection From a Vertical Plate With Uniform Surface Heat Flux," *Trans. ASME,* Vol. 78, pp. 435-440.

Sparrow, E. M., and Gregg, J. L., 1958, "Similar Solutions for Free Convection From a Nonisothermal Vertical Plate," *Trans. ASME,* Vol. 80, pp. 379-386.

Sparrow, E. M., and Lin, S. H., 1965, "Boundary Layers With Prescribed Heat Flux - Application To Simultaneous Convection and Radiation," *Int. J. Heat Mass Transfer,* Vol. 8, pp. 437-448.

Yang, K. T., 1960, "Possible Similarity Solutions for Laminar Free Convection on Vertical Plates and Cylinders," ASME *Journal of Applied Mechanics,* Vol. 82, pp. 230-236.

Yang, J., Jeng, D. R., and De Witt K. J., 1982, "Laminar Free Convection From a Vertical Plate With Nonuniform Surface Conditions," *Numerical Heat Transfer,* Vol. 5, pp. 165-184.