# NEW NUSSELT AND SHERWOOD NUMBERS FOR ARBITRARY ISOPOTENTIAL BODIES AT NEAR ZERO PECLET AND RAYLEIGH NUMBERS

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Abstract

A general expression is developed for the dimensionless numbers: Nusselt and Sherwood under near zero forced and free convection conditions. The expression is valid for isopotential ellipsoids in an extensive stagnant fluid. Special cases which arise from the general expression are: oblate and prolate spheroids; spheres; elliptical and circular disks; and long elliptical and circular cylinders. A new characteristic length based on the square root of the total surface area is introduced and used in a new general expression for the Nusselt and Sherwood numbers which are shown to be weak functions of the ellipsoid shape and aspect ratio over a wide range of these parameters. The analytical expressions for the special cases can be used to approximate the results for many other *similar* bodies for which analytical, experimental, and numerical solutions are presently not available.

Nomenclature

| | |
|---|---|
| $A$ | surface area of the body |
| $AR$ | aspect ratio of oblate and prolate spheroids |
| $\mathcal{A}$ | dimensionless surface area |
| $B$ | correlation coefficient |
| $a, b, c$ | semi-axes of the ellipsoidal body |
| $C$ | capacitance of a charged body $[Q/V_0]$ |
| $C_{\mathcal{L}}^*$ | dimensionless capacitance $[C_{\mathcal{L}}^* = C\mathcal{L}/\epsilon A]$ |
| $c$ | concentration in the extensive fluid |
| $c_0$ | uniform body concentration |
| $c_\infty$ | concentration remote from body |
| $D$ | body diameter |
| $\mathcal{D}$ | molecular diffusivity |
| $E(\phi, \kappa)$ | incomplete elliptic integral of the second kind |
| $e$ | eccentricity of the oblate and prolate spheroids |
| $F(\kappa, \theta)$ | incomplete elliptic integral of the first kind |
| $f_o$ | approximation for oblate area |
| $f_p$ | approximation for prolate area |
| $g$ | scalar gravitational acceleration |
| $Gr_{\mathcal{L}}$ | Grashof number $[g\beta(T_0 - T_\infty)\mathcal{L}^3/\nu^2]$ |
| $h$ | heat transfer coefficient |
| $h_m$ | mass transfer coefficient |
| $K(\kappa)$ | complete elliptic integral of the first kind |
| $k$ | thermal conductivity of the extensive fluid |
| $L$ | body length |
| $\mathcal{L}$ | characteristic length of the body |
| $\dot{m}$ | mass flow rate |
| $Nu_{\mathcal{L}}$ | Nusselt number $[Nu_{\mathcal{L}} = h\mathcal{L}/k]$ |
| $n$ | outward body normal |
| $\tilde{n}$ | dimensionless outward body normal |
| $Pe$ | Peclet number $[RePr]$ |
| $Pr$ | Prandtl number $[\nu/\alpha]$ |
| $p$ | pressure |
| $Q$ | heat flow rate; charge on body |
| $R$ | diffusive resistance of the extensive fluid |
| $R_{\mathcal{L}}^*$ | dimensionless resistance $[R_{\mathcal{L}}^* = AkR/\mathcal{L}]$ |
| $Ra_{\mathcal{L}}$ | Rayleigh number $[Gr_{\mathcal{L}}Pr = g\beta(T_0 - T_\infty)\mathcal{L}^3/\nu\alpha]$ |
| $Re_{\mathcal{L}}$ | Reynolds number $[U\mathcal{L}/\nu]$ |
| $r$ | position vector |
| $S$ | shape factor |
| $S^*$ | dimensionless shape factor $[S\mathcal{L}/A]$ |
| $Sc$ | Schmidt number $[\nu/\mathcal{D}]$ |
| $Sh_{\mathcal{L}}$ | Sherwood number $[Sh_{\mathcal{L}} = h_m\mathcal{L}/\mathcal{D}]$ |
| $T$ | temperature |
| $T_0$ | uniform body temperature |
| $T_\infty$ | fluid temperature remote from the body |
| $t$ | time |
| $x, y, z$ | cartesian coordinates |
| $v_\infty$ | reference velocity |
| $V_0$ | voltage of charged body |
| $\vec{V}$ | velocity vector |
| $\tilde{\mathbf{V}}$ | dimensionless velocity vector |

Greek Letters

| | |
|---|---|
| $\alpha$ | thermal diffusivity |
| $\beta$ | thermal compressibility coefficient |
| $\epsilon$ | permittivity of space |
| $\theta$ | amplitude of incomplete elliptic integral |
| $\kappa$ | modulus of elliptic integrals |
| $\mu$ | ellipsoidal coordinate |
| $\nu$ | kinematic viscosity of the extensive fluid |
| $\rho$ | mass density of the extensive fluid |
| $\rho_\infty$ | density of fluid remote from the body |
| $\tau$ | dimensionless time |
| $\phi$ | dimensionless potential in the extensive fluid |
| $\psi$ | dimensionless resistance |

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## Introduction

Many researchers over several decades have published hundreds of theoretical, experimental and numerical papers which consider heat and mass transfer from isopotential (isothermal or isoconcentration) bodies of arbitrary shape under free and forced convection into a stagnant fluid of large extent. The results in heat transfer are reported as Nusselt number, Nu, and in mass transfer as Sherwood number, Sh, plotted and/or correlated against the Peclet number, Pe (RePr or ReSc), in forced flow or the Rayleigh number, Ra (GrPr or GrSc), in natural convection. Often correlations of the theoretical and experimental results are given for a particular range of the Peclet or Rayleigh numbers; and over the years a large number of these correlations have been developed and they now appear in handbooks and in the many heat transfer texts [1-11] presently available to the interested researcher, engineer and student.

With the exception of vertical plates [12,21,22,23], circular cylinders [13-20], spheres [24,25], bisphere [26] and cubes in several orientations [26,27], the various correlations do not include the near zero Peclet and Rayleigh number range which is of considerable interest to those thermal analysts who must deal with i) small bodies ii) small temperature or concentration differences iii) creeping flow or iv) miro-gravity effects such as in aerospace applications. There are also other conditions which can result in very low to near zero Peclet and Rayleigh numbers in both heat and mass transfer cases.

There is, therefore, a great need to extend the existing correlations to the near zero Peclet or Rayleigh number range and to develop new correlations for the many geometries which are of interest to the thermophysics and heat transfer community.

The purpose of the paper is to obtain a general analytical solution for a body shape which contains many other body shapes as special cases. Figure 1 shows how the ellipsoid is related to several body shapes such as oblate and prolate spheroids, spheres, elliptical and circular disks, and the elliptical and circular cylinders. The ellipsoid is obviously a general body shape whose solution should reduce to the solutions for the particular body shapes discussed above. The solutions for the special cases can then be used to approximate the heat and mass from bodies of arbitrary shape for which solutions are not available.

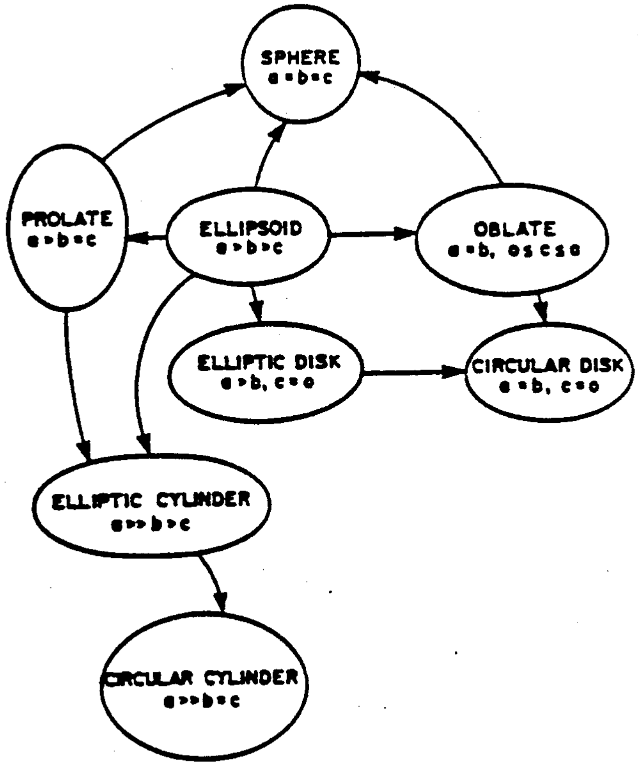



Fig.1 Ellipsoid and special body shapes

## General Field Problem Arising From Forced and Natural Convection Heat and Mass Transfer for $Pe_\ell$ and $Ra_\ell$ Approaching Zero

Forced and natural convection heat and mass transfer from isothermal or isoconcentration bodies of arbitrary shape are similar mathematical problems when the governing equations and boundary conditions are nondimensionalized. In the limit of zero forced or natural convection, i.e., pure diffusive heat and mass transfer, the thermal and mass transfer problems are similar to the classical capacitance problem. In this section the dimensionless groups from convective heat and mass transfer will be related to the conduction and capacitance dimensionless groups.

### Convective Heat Transfer

Application of the first law of thermodynamics to a differential control volume of incompressible Newtonian fluid having constant properties, and neglecting the viscous dissipation term yields the simple form of the energy equation

$$\frac{DT}{Dt} = \alpha \nabla^2 T \qquad (1)$$

where $\alpha$ is the thermal diffusivity of the fluid.

The term on the left-hand side can be expanded using the expression for the substantial derivative, or Stokes operator,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V} \cdot \nabla \qquad (2)$$

where the first term, called the local derivative, represents changes at a fixed point in the fluid and the second term, the convective term, accounts for changes following the motion of the fluid.

The velocity vector appearing in the substantial derivative is the solution of the Navier-Stokes equation of motion

$$\rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p + \mu\nabla^2\vec{V} \quad (3)$$

which is obtained from the application of Newton's second law of motion to a differential control volume of an incompressible Newtonian fluid of density $\rho$ and constant viscosity $\mu$, under the action of gravity as the only body force. The remaining two terms in Eq. (3) are the pressure and viscous forces which act normal and tangential to the control volume surfaces.

Application of the principle of conservation of mass to an incompressible fluid yields the continuity equation:

$$\nabla\cdot\vec{V} = 0 \quad (4)$$

These equations are subject to the following boundary conditions on the temperature and the velocity fields:

i) on the body:

$$T = T_0 \quad \text{and} \quad \vec{V} = 0 \quad (5)$$

ii) remote from the body:

$$T = T_\infty \quad \text{and} \quad \vec{V} = 0 \quad \text{or} \quad \vec{V} = \vec{U}_\infty \quad (6)$$

It is convenient to introduce the dimensionless temperature difference or potential difference

$$\phi = \frac{(T - T_\infty)}{(T_0 - T_\infty)} \quad (7)$$

which has the values unity and zero on the body and at points remote from the body respectively.

The no-slip condition is used on the body and the zero velocity or uniform velocity conditions are used for the natural convection and forced convection problems respectively.

The area-mean Nusselt number is obtained from the application of Newton's cooling law and Fourier's law of conduction to the fluid at the body surface:

$$Q = \iint_A h(\mathbf{r})(T_0 - T_\infty)dA = \iint_A -k\left(\frac{\partial T}{\partial n}\right)_0 dA \quad (8)$$

where $h(\mathbf{r})$ is the local value of the heat transfer coefficient, $\mathbf{r}$ is the position vector, $k$ is the fluid thermal conductivity, $(\partial T/\partial n)$ is the local temperature gradient normal to the body surface and $dA$ is the differential heat transfer area on the body.

Introducing the area-mean Nusselt number

$$Nu_{\mathcal{L}} = \frac{h\mathcal{L}}{k} \quad (9)$$

where $h$ is the area-mean heat transfer coefficient defined as

$$h = \frac{1}{A}\iint_A h(\mathbf{r})dA \quad (10)$$

and $\mathcal{L}$ is some characteristic length of the body, yields the following relationships:

$$Nu_{\mathcal{L}} = \frac{\mathcal{L}}{kA}\iint_A h(\mathbf{r})dA = \frac{\mathcal{L}}{A}\iint_A -\left(\frac{\partial\phi}{\partial n}\right)_0 dA \quad (11)$$

and

$$Nu_{\mathcal{L}} = Q^* = \frac{Q\mathcal{L}}{A(T_0 - T_\infty)k} = \frac{1}{R^*_{\mathcal{L}}} \quad (12)$$

where $Q^*$ is the dimensionless heat flow rate and $R^*_{\mathcal{L}}$ is the dimensionless overall thermal resistance defined as $R = (T_0 - T_\infty)/Q$. To obtain the area-mean Nusselt number it is necessary to obtain solutions to the dimensionless energy equation

$$\frac{D\phi}{D\tau} = \frac{1}{Pe_{\mathcal{L}}}\tilde{\nabla}^2\phi \quad (13)$$

and the dimensionless Navier-Stokes equation

$$\frac{D\tilde{V}}{D\tau} = -\tilde{\nabla}\tilde{p} + \frac{1}{Re_{\mathcal{L}}}\tilde{\nabla}^2\tilde{V} \quad (14)$$

where the tildes denote dimensionless quantities or operators formed using dimensionless variables. The Peclet number which appears in the energy equation is $Pe_{\mathcal{L}} = Re_{\mathcal{L}}Pr$. Reference quantities $U_\infty$, $p_\infty$ and the characteristic length $\mathcal{L}$ are used with the fluid thermophysical and transport properties to form the following dimensionless quantities and groups:

dimensionless position and position vector:

$$\tilde{x} = \frac{x}{\mathcal{L}}, \quad \tilde{y} = \frac{y}{\mathcal{L}}, \quad \tilde{z} = \frac{z}{\mathcal{L}}, \quad \tilde{\mathbf{r}} = \frac{\mathbf{r}}{\mathcal{L}} \quad (15)$$

dimensionless surface area and outward normal:

$$\tilde{A} = \frac{A}{\mathcal{L}^2}, \quad \tilde{n} = \frac{n}{\mathcal{L}} \quad (16)$$

dimensionless velocity vector:

$$\tilde{V} = \frac{\vec{V}}{U_\infty} \quad (17)$$

dimensionless time:

$$\tau = \frac{t\,U_\infty}{\mathcal{L}} \quad (18)$$

dimensionless pressure

$$\tilde{p} = \frac{(p - p_\infty)}{\rho\,U_\infty^2} \quad (19)$$

and the Reynolds number:

$$Re_{\mathcal{L}} = \frac{U_\infty\mathcal{L}}{\nu} \quad (20)$$

Natural convection heat transfer is produced by density gradients which are due to temperature gradients, and because the body force is caused by the density gradients the energy and momentum equations are coupled and the solutions are difficult to obtain. A set of simplifying assumptions, called the Boussinesq approximation, is used to obtain the solutions. Analyses of natural convection assume that:

1. density is constant in the continuity and momentum equations, except in the body force;

2. density variations are caused by temperature gradients;

3. all other thermophysical properties are constant.

The variable mass density $\rho$ is expanded in a Taylor series about the density of the fluid remote from the body, $\rho_\infty$:

$$\rho = \rho_\infty + (T - T_\infty)\left(\frac{\partial \rho}{\partial T}\right)_{\rho_\infty} \tag{21}$$

Introducing the compressibilty coefficient

$$\beta = -\frac{1}{\rho_\infty}\left(\frac{\partial \rho}{\partial T}\right)_{\rho_\infty} \tag{22}$$

and the assumptions given above, the steady-state momentum equation becomes

$$\vec{V}\cdot\nabla\vec{V} = -\frac{\nabla p}{\rho_\infty} + \beta(T - T_\infty)\vec{g} + \nu\nabla^2\vec{V} \tag{23}$$

which can be transformed into dimensionless form:

$$\tilde{\mathcal{V}}\cdot\tilde{\nabla}\tilde{\mathcal{V}} = -\tilde{\nabla}\tilde{p} - \frac{Gr_\mathcal{L}}{Re_\mathcal{L}^2}\phi\frac{\vec{g}}{g} + \frac{1}{Re_\mathcal{L}}\tilde{\nabla}^2\tilde{\mathcal{V}} \tag{24}$$

where $Gr_\mathcal{L} = g\beta(T_0 - T_\infty)\mathcal{L}^3/\nu^2$ is the Grashof number, an important dimensionless group in natural convection heat transfer.

The continuity, momentum and energy equations for steady-state, constant properties and $Pe_\mathcal{L} \to 0$ for forced convection heat transfer and $Ra_\mathcal{L} \to 0$ for natural convection reduce to the following equations:

$$\tilde{\nabla}\cdot\tilde{\mathcal{V}} = 0 \tag{25}$$

$$Pr\,\tilde{\nabla}^2\tilde{\mathcal{V}} = 0 \tag{26}$$

$$\tilde{\nabla}^2\phi = 0 \tag{27}$$

The Nusselt number is to be determined by the following dimensionless equation:

$$Nu_\mathcal{L} = \frac{1}{\tilde{A}}\iint_{\tilde{A}} -\left(\frac{\partial \phi}{\partial \tilde{n}}\right)_0 d\tilde{A} \tag{28}$$

Mixed Convection $Gr_\mathcal{L} > 0, Re_\mathcal{L} > 0$

For mixed convection heat transfer the Nusselt number is expected to depend on several dimensionless parameters such that

$$Nu_\mathcal{L} = f(Gr_\mathcal{L}, Re_\mathcal{L}, Pr, \mathcal{L}, AR) \tag{29}$$

where $AR$ is the body aspect ratio defined as the maximum body length parallel to the gravity vector divided by the maximum body length perpendicular to the gravity vector.

Forced Convection $Gr_\mathcal{L} > 0, Re_\mathcal{L} > 0, Gr_\mathcal{L}/Re_\mathcal{L}^2 << 1$

For pure forced convection heat transfer the Nusselt number is expected to depend on several dimensionless parameters such that

$$Nu_\mathcal{L} = f(Re_\mathcal{L}, Pr, \mathcal{L}, AR) \tag{30}$$

Natural Convection $Gr_\mathcal{L} > 0, Re_\mathcal{L} > 0, Gr_\mathcal{L}/Re_\mathcal{L}^2 >> 1$

For pure natural convection heat transfer the Nusselt number is expected to depend on several dimensionless parameters such that

$$Nu_\mathcal{L} = f(Gr_\mathcal{L}, Pr, \mathcal{L}, AR) \tag{31}$$

Conduction Heat Transfer $Gr_\mathcal{L} \to 0, Re_\mathcal{L} \to 0$

For pure conduction heat transfer the Nusselt number is expected to depend on the geometry of the body only such that

$$Nu_\mathcal{L} = f(\mathcal{L}, AR) \tag{32}$$

It is anticipated that the proper choice of the characteristic length $\mathcal{L}$ will minimize the dependence of the Nusselt number on the independent parameters, and this will be demonstrated for pure conduction from isothermal ellipsoids.

Convective Mass Transfer

Application of the principle of conservation of mass to a binary system consisting of a non-reactive solute in dilute solution in an incompressible fluid yields

$$\frac{Dc}{Dt} = \mathcal{D}\nabla^2 c \tag{33}$$

where $\mathcal{D}$, the diffusivity, is assumed constant. The driving force for diffusion is provided by molar concentration gradients, and therefore, Eq. (33) is an adequate description of diffusion in most liquids, since the density is essentially constant, and in gases when the molecular weight of the solute is similar to that of the host gas.

If the solution is dilute, the diffusion process will not appreciably alter the fluid motion, so that the velocity field can be considered to be unaltered and the momentum and continuity equations discussed under convection heat transfer also apply here. The Reynolds number is the same, but the Peclet number is now defined as $Pe_\mathcal{L} = Re_\mathcal{L}Sc$ where the Schmidt number replaces the Prandtl number in forced convection. The Schmidt number is $Sc = \nu/\mathcal{D}$.

The dimensionless concentration is defined as

$$\phi = \frac{(c - c_\infty)}{(c_0 - c_\infty)} \tag{34}$$

where $c_0$ and $c_\infty$ are the concentrations on the body and at points remote from the body respectively.

The area-mean Sherwood number, $Sh$, is obtained from the application of the mass transfer relationship and Fick's law of mass transfer to the fluid at the body surface:

$$\dot{m} = \iint_A h_m(r)(c_0 - c_\infty)dA = \iint_A -\mathcal{D}\left(\frac{\partial c}{\partial n}\right)_0 dA \tag{35}$$

where $h_m(\mathbf{r})$ is the local mass transfer coefficient analogous to the heat transfer coefficient.

Introducing the area-mean Sherwood number

$$Sh_{\mathcal{L}} = \frac{h_m \mathcal{L}}{D} \tag{36}$$

where $h_m$ is defined by Eq. (35) in which $h(\mathbf{r})$ is replaced by $h_m(\mathbf{r})$, yields the following dimensionless relationship:

$$Sh_{\mathcal{L}} = \frac{\mathcal{L}}{A} \iint_A -\left(\frac{\partial \phi}{\partial \bar{n}}\right)_0 dA \tag{37}$$

and thus we see that $Sh_{\mathcal{L}} = Nu_{\mathcal{L}}$ for the same flow conditions which are determined by the Peclet number and the Grashof number which for convective mass transfer is defined as

$$Gr_{\mathcal{L}} = \frac{g\beta_c(c_0 - c_\infty)\mathcal{L}^3}{\nu^2} \tag{38}$$

where

$$\beta_c = -\frac{1}{\rho_\infty}\left(\frac{\partial \rho}{\partial c}\right)_{p_\infty} \tag{39}$$

Thus it is seen that the determination of the area-mean Sherwood number is identical to the determination of the area-mean Nusselt number, and the functional dependence of the Sherwood number on the several physical and geometric parameters is also identical.

## Capacitance of Isopotential Bodies in Free Space

The diffusion of heat and mass from an isopotential body into a stagnant fluid of large extent is analogous to the determination of the total charge supported by an isopotential body which is charged to some voltage above ground voltage when it is located in a charge-free homogeneous dielectric medium whose permittivity is $\epsilon$. The voltage must satisfy the Laplace equation:

$$\nabla^2 V = 0 \tag{40}$$

and the total charge $Q$ is determined from

$$Q = \iint_A -\epsilon \frac{\partial V}{\partial n} dA \tag{41}$$

The capacitance $C = Q/V_0$ where $V_0$ is the body voltage can be nondimensionlized

$$C^*_{\mathcal{L}} = \frac{C\mathcal{L}}{\epsilon A} = \frac{1}{A} \iint_A -\left(\frac{\partial \phi}{\partial \bar{n}}\right) dA \tag{42}$$

which is seen to be identical to the equations developed for the convective heat and mass transfer problems.

## General Relationship Between Various Dimensionless Heat, Mass, Conduction and Capacitance Groups

We can now write the general relationship which brings together the heat and mass transfer dimensionless groups, the pure conduction groups and the capacitance group:

$$Nu_{\mathcal{L}} = Sh_{\mathcal{L}} = C^*_{\mathcal{L}} = \frac{1}{R^*_{\mathcal{L}}} = S^* \tag{43}$$

where $S^* = S\mathcal{L}/A$ is the dimensionless conduction shape factor which is defined as [34,42,43]:

$$Q = kS(T_0 - T_\infty) \tag{44}$$

Thus it can be seen that the conduction shape factor, $S$, and the conductive resistance, $R$, are related:

$$S = \frac{1}{kR} \tag{45}$$

By means of these relationships it is possible to determine the geometric dependence of one group given the geometric dependence of some other group.

## Ellipsoidal Problem and Its Solution

Consider an isothermal ellipsoid with semi-axes $a \geq b \geq c \geq 0$ oriented along the $x, y$ and $z-$ axes, respectively, with the origin of the Cartesian coordinate system placed at the centroid of the ellipsoid.

When steady-state heat transfer from the isothermal ellipsoid into the extensive fluid occurs under the conditions that both Peclet and Rayleigh numbers are zero or nearly zero, the dimensionless temperature field within the fluid must satisfy Laplace's equation [41]:

$$\nabla^2 \phi = \frac{d}{d\mu}\left[\sqrt{(a^2+\mu)(b^2+\mu)(c^2+\mu)}\,\frac{d\phi}{d\mu}\right] = 0 \tag{46}$$

where the dimensionless temperature has been defined as

$$\phi = \frac{T - T_\infty}{T_0 - T_\infty} \tag{47}$$

and the ellipsoidal parameter $\mu$ is the positive root of

$$\frac{x^2}{a^2+\mu} + \frac{y^2}{b^2+\mu} + \frac{z^2}{c^2+\mu} = 1 \tag{48}$$

The dimensionless temperature (isopotential) must also satisfy the following uniform Dirichlet boundary conditions:

$$\phi = 1 \quad \text{when} \quad \mu = 0 \tag{49}$$

$$\phi = 0 \quad \text{when} \quad \mu = \infty \tag{50}$$

The solution of Eq.(46) is [41]:

$$\phi = \frac{F(\kappa, \theta_\mu)}{F(\kappa, \theta)} \qquad 0 \leq \mu \leq \infty \tag{51}$$

with

$$\kappa = \sqrt{\frac{a^2 - b^2}{a^2 - c^2}} \tag{52}$$

$$\sin\theta_\mu = \sqrt{\frac{a^2 - c^2}{a^2 + \mu}} \tag{53}$$

$$\sin\theta = \sqrt{\frac{a^2 - c^2}{a^2}} \tag{54}$$

The functions which appear in the solution are incomplete elliptic integrals of the first kind of modulus $\kappa$ and amplitude angles $\theta$ and $\theta_\mu$.

The isopotentials in the neighborhood of the body are ellipsoidal and at points remote from the body they are nearly spherical. The complex isopotential distribution within the extensive fluid can be described in terms of the single ellipsoidal parameter $\mu$.

The total heat flow rate through the extensive fluid can be obtained by means of a heat balance over the total surface of the isothermal body or more easily over a control surface located at points which are at large distances relative to the largest length of the body. Selecting a spherical control surface whose radius is orders of magnitude larger than the semimajor axis of the ellipsoid, the heat balance gives [41]:

$$Q = \frac{8\pi k(T_0 - T_\infty)}{\int_0^\infty \frac{d\mu}{\sqrt{(a^2+\mu)(b^2+\mu)(c^2+\mu)}}} \tag{55}$$

or equivalently, in terms of the incomplete elliptic integral of the first kind:

$$Q = \frac{4\pi k(T_0 - T_\infty)\sqrt{a^2 - c^2}}{F(\kappa, \theta)} \tag{56}$$

From the last equation the dimensionless resistance, $R^*_{\mathcal{L}}$, or the diffusive Nusselt number, $Nu_{\mathcal{L}}$, are obtained directly,

$$R^*_{\mathcal{L}} = \frac{\mathcal{L}F(\kappa, \theta)}{4\pi\sqrt{a^2 - c^2}} \tag{57}$$

and

$$Nu_{\mathcal{L}} = \frac{4\pi\sqrt{a^2 - c^2}\,\mathcal{L}}{AF(\kappa, \theta)} \tag{58}$$

The appropriate choice of the characteristic body length will be dealt with in a later section after discussions of surface areas and intrinsic resistances.

## Surface Area of Ellipsoids and Disks

The surface area of an ellipsoid having semiaxes $a \geq b \geq c$ is

$$A = 2\pi c^2 + \frac{2\pi ab}{\sin\phi}\left[\left(\frac{c}{a}\right)^2 F(\phi, \kappa) + \left(\frac{a^2 - c^2}{a^2}\right) E(\phi, \kappa)\right] \tag{59}$$

with

$$\cos\phi = \frac{c}{a} \quad \text{and} \quad \kappa^2 = \frac{b^2 - c^2}{b^2 \sin^2\phi} \tag{60}$$

The special functions which appear in Eq. (59) are incomplete elliptic integrals of the first and second kind of amplitude $\phi$ and modulus $\kappa$.

Several special cases arise out of this general case:

1. Oblate Spheroids: $a = b \geq c$. The surface area is

$$A = 2\pi a^2\left[1 + \frac{1 - e^2}{e}\frac{1}{2}\ln\frac{1 + e}{1 - e}\right] \tag{61}$$

with $e = \sqrt{1 - (c/a)^2}$.

2. Prolate Spheroids: $a \geq b = c$. The surface area is

$$A = 2\pi b^2\left[1 + \frac{\sin^{-1} e}{e\sqrt{1 - e^2}}\right] \tag{62}$$

with $e = \sqrt{1 - (b/a)^2}$.

3. Sphere: $a = b = c$. The surface area is

$$A = 4\pi a^2 \tag{63}$$

4. Elliptical Disks: $a \geq b$ and $c = 0$. The total area is

$$A = 2\pi ab \tag{64}$$

5. Circular Disk: $a = b, c = 0$. The total area is

$$A = 2\pi a^2 \tag{65}$$

It should be noted that both sides of the disks are considered because the heat and mass transfer are assumed to occur at both surfaces.

## Dimensionless Diffusive Resistances

The dimensionless diffusive resistance $R^* = AkR/\mathcal{L}$ is equivalent to the reciprocal of the dimensionless Nusselt and Sherwood numbers. For the ellipsoids and the special cases which arise from the general solution the dimensionless resistances are:

1. Ellipsoids: $a \geq b \geq c$.

$$\sqrt{a^2 - c^2}\,kR = \frac{1}{4\pi}F\left(\arcsin\sqrt{\frac{a^2 - c^2}{a^2}},\ \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}\right) \tag{66}$$

The special function $F(\phi, \kappa)$ which appears in the ellipsoidal resistance is the incomplete elliptic integral of the first kind of amplitude $\phi$ and modulus $\kappa$ as they are defined in Eq. (66).

2. Prolate Spheroids: $a \geq b = c$.

$$\sqrt{a^2 - c^2}\,kR = \frac{1}{8\pi}\ln\left[\frac{a + \sqrt{a^2 - c^2}}{a - \sqrt{a^2 - c^2}}\right] \tag{67}$$

3. Oblate Spheroids: $a = b, 0 \leq c \leq a$.

$$\sqrt{a^2 - c^2}\,kR = \frac{1}{4\pi}\cos^{-1}\left(\frac{c}{a}\right) \tag{68}$$

4. Spheres: $a = b = c$.

$$akR = \frac{1}{4\pi} \tag{69}$$

5. Elliptical Disks: $a \geq b, c = 0$.

$$akR = \frac{1}{4\pi}K\left(\sqrt{\frac{a^2 - b^2}{a^2}}\right) \tag{70}$$

The function appearing in Eq. (70) is the complete elliptic integral of the first kind of modulus $\kappa = \sqrt{(a^2 - b^2)/a^2}$. Polynomial approximations are available for accurate computation of $K(\kappa)$ [44].

6. Circular Disks: $a = b, c = 0$.

$$akR = \frac{1}{8} \tag{71}$$

The characteristic lengths used to nondimensionalize the conductive resistances are those which arise naturally from the analysis. For the ellipsoid and the two spheroids the characteristic or intrinsic length is seen to be equal to the square root of the difference between the squares of the largest and smallest semiaxes. For the sphere and the circular disk the intrinsic length is seen to be the radius, and in the elliptical disk it is the semimajor axis.

With the exception of the sphere and circular disk the dimensionless resistance is dependent on the ratio of the smallest and largest semiaxis or the body aspect ratio. The body aspect ratio for the oblate spheroid is $u = c/a$ where $0 \leq u \leq 1$. The body aspect ratio for the prolate spheroid can be defined as $c/a$ or $a/c$, but to be consistent with the oblate spheroid for which the body aspect ratio is the semiaxis parallel to the bulk flow divided by the semiaxis perpendicular to the flow, we will let $u = a/c$. It can be seen that the ellipsoid has two body aspect ratios $b/a$ and $c/a$. The sphere has unit body aspect ratio and the disks have zero body aspect ratio.

## Characteristic Lengths Used to Correlate Forced and Free Convection Heat and Mass Transfer From Spheroids

Various characteristic lengths have been used to nondimensionalize the Nusselt and Sherwood numbers under forced and free convection conditions. In this section the characteristic lengths which have been used in heat and mass transfer will be discussed with respect to the oblate spheroid which is axisymmetric about its minor axis and has two axes one of which goes to zero in the limit when the oblate spheroid becomes a circular disk having zero thickness.

In the discussion which follows it is assumed that when forced or free convection occurs, the bulk flow of the surrounding fluid takes place parallel to the minor axis only. The semiminor and semimajor axes are denoted by $c$ and $b$ respectively. The total surface area and the volume are denoted by $A$ and $V$ respectively. The maximum perimeter of the spheroid is denoted by $P_{max}$ and it lies in the plane of the major axis. There is another perimeter which lies in the plane which contains the minor axis. It is associated with the boundary layer flow length.

1. Length of the axis parallel to the motion of the fluid. For the oblate the length is the minor axis, $\mathcal{L}_1 = 2c$.

2. Length of the axis perpendicular to the motion of the fluid. For the oblate the length is the major axis, $\mathcal{L}_2 = 2b$.

3. The arithmetic mean of the axes parallel and perpendicular to the motion of the fluid. For the oblate the length is the arithmetic mean of the minor and major axes, $\mathcal{L}_3 = (c + b)$.

4. Length is the geometric mean of the semiaxes. $\mathcal{L}_4 = \sqrt{bc}$.

5. Length is the harmonic mean of the semiaxes. $1/\mathcal{L}_5 = (1/b + 1/c)/2$.

6. Length is the volume of the body divided by the total surface area of the body. $\mathcal{L}_6 = V/A$.

7. Length is the cube root of the volume of the body. $\mathcal{L}_7 = V^{\frac{1}{3}}$.

8. Length is the square root of the total surface area of the body. $\mathcal{L}_8 = \sqrt{A}$.

9. Length is the total surface area of the body divided by the perimeter of the maximum projected area normal to the motion of the fluid. $\mathcal{L}_9 = A/P_{max}$.

10. Length is the diameter of a sphere having the same total surface area of the body. $\mathcal{L}_{10} = \sqrt{A/\pi}$.

11. Length is the diameter of a sphere having the same volume as the body. $\mathcal{L}_{11} = (6V/\pi)^{\frac{1}{3}}$.

12. Length is the sphericity multiplied by the diameter of a sphere having the same volume as the body.

Other more complex definitions of the characteristic length are available in the literature. The ones which are of interest here are those recently examined by Sparrow and Stretton [27] and Sparrow and Ansari [28].

13. King's length is defined as $1/\mathcal{L}_K = 1/\mathcal{L}_v + 1/\mathcal{L}_h$ where $\mathcal{L}_v$ and $\mathcal{L}_h$ are the minor and major axes of the oblate spheroid. In this case King's length is the harmonic mean length, $\mathcal{L}_K = \mathcal{L}_5$.

14. Lienhard's length is defined to be equal to the length of travel of the fluid in the boundary layer. For the case of the oblate spheroid the Lienhard length is equal to half the perimeter of the elliptical cross section, and therefore $\mathcal{L}_H = 2bE(\kappa)$ where $E(\kappa)$ is the complete elliptic integral of the second kind of modulus $\kappa = \sqrt{1 - (c/b)^2}$.

15. Sparrow-Stretton length [27] is defined to be equal to the total surface area of the body divided by the equivalent diameter of a circular area set equal to the area of the projection of the body onto a plane situated either above or below the body. For the oblate spheroid the Sparrow-Stretton length is equal to its area divided by its perimeter in the plane perpendicular to the minor axis multiplied by the constant $\pi$. Therefore, $\mathcal{L}_{SS} = \pi A/P_{max}$.

The oblate spheroid is an interesting and important body because in one limit it is a sphere having unit aspect ratio, and in the other limit it becomes a circular disk having zero thickness, zero volume and zero body aspect ratio. If the circular disk is included in the development of heat and mass transfer correlation equations, then any characteristic length based upon the minor axis or containing it in some combination which goes to zero or becomes unbounded cannot be used. For this reason $\mathcal{L}_1, \mathcal{L}_4, \mathcal{L}_5, \mathcal{L}_6, \mathcal{L}_7, \mathcal{L}_{11}$, and $\mathcal{L}_K$ cannot be used to nondimensionalize the oblate spheroidal results.

Sparrow and Ansari [28] obtained free convection results for a vertical short cylinder(length = diameter) in the Rayleigh number range of $3 \times 10^3$ to $1.5 \times 10^5$ when the diameter of the cylinder is used as the characteristic length. They examined King's length, $\mathcal{L}_K = D/2$, and Lienhard's length, $\mathcal{L}_L = 2D$, with respect to their experimental results. They concluded that King's length which led to 40 − 50% overpredictions of the experimental data should not be used in free convection correlations. They observed that Lienhard's length was somewhat better because at the lower end of the Rayleigh number range Lienhard's length underpredicted the experimental results by about 30%, but at the higher end the predictions were within 8% of the experimental results. They further concluded that Lienhard's length does not warrant its adoption as the link between multi-dimensional free convection and the established literature.

Sparrow and Stretton [27] conducted a series of free convection experiments from an isothermal cube (aspect ratio =1) in various orientations which included the horizontal top/horizontal bottom case, the on edge case and the diagonal aligned case. The steady-state experiments were conducted in air and in water and the combined Rayleigh number based on the side of the cube ranged from $2 \times 10^3$ to $10^7$. The King's length is equal to half the cube side, that is, $\mathcal{L}_K = S/2$, and it overpredicted the experimental results by 40 − 58%, while Lienhard's length which ranged from 2 to 2.414 times the cube side underpredicted the experimental results by as much as 23%. As a result of these observations, they introduced their own characteristic length which for the cube in the horizontal top/horizontal bottom orientation was found to be $\mathcal{L}_{SS} = 3S/\sqrt{\pi}$. They found it gave a very tight fit to their experimental data, which for all cube orientations were fitted with the correlation equation

$$Nu_{SS} = 6.65 + 0.623\,[Ra_{SS}/F(Pr)]^{0.261} \qquad (72)$$

where the function

$$F(Pr) = \left[1 + (0.492/Pr)^{9/16}\right]^{16/9} \qquad (73)$$

was highly effective in correlating the air and water data. They found that 78% of the data fall within 2% of the correlating equation and that the maximum deviation is only 5.5%.

The constant 6.65 was determined empirically and it is the limiting value of $Nu_{SS}$ as $Ra_{SS} \to 0$. This pure conduction limit can be compared with the calculated value $Nu_S = 1.383$ which was based on the cube side $S$. They concluded that since $\mathcal{L}_{SS}/S$ ranges from 5.317 to 4.040 corresponding to the extreme orientations, the conduction limit for $Nu_{SS}$ should range from 5.587 to 7.353, which bounds the empirical value of 6.65.

The apparent success of the $Nu_{SS}$ and $Ra_{SS}$ groups in correlating the cube data prompted Sparrow and Stretton to use these groups for other bodies of unity aspect ratio. The available experimental data for spheres in air, spheres in water, short cylinder in air, and cubes in air were recast in terms of $Nu_{SS}$ and $Ra_{SS}$ and correlated by the equation

$$Nu_{SS} = 5.748 + 0.752\,[Ra_{SS}/F(Pr)]^{0.252} \qquad (74)$$

where 84% of the data fell within 5% of the correlation, and the maximum deviation is 9.5%. The Rayleigh number ranged from about 200 to $1.5 \times 10^9$.

When Sparrow and Stretton used their correlation to predict free convection from oblate spheroids of aspect ratios of 0.1 and 0.5, and a prolate spheroid of aspect ratio of 1.93, and the vertically aligned bi-sphere, they observed significant differences, especially at the lower values of the Rayleigh number. They concluded that their general correlation can be used with confidence for bodies having aspect ratios which deviate slightly from unity, but it is not recommended for bodies whose aspect ratios deviate significantly from unity.

## General Dimensionless Resistance Expression

The general expression for the conductive resistance can be obtained from the general solution of the resistance by multiplying the result by the new characteristic length, that is, the square root of the total body area $\sqrt{A}$. Using the expression developed in the previous section we obtain

$$\begin{aligned} R^*_{\sqrt{A}} &= k\sqrt{A}R \\ &= \frac{1}{\sqrt{8\pi}}\left[\frac{1}{\tan^2\phi} + \frac{v}{\sin\phi}\left[\frac{F(\phi,\kappa)}{\tan^2\phi} + E(\phi,\kappa)\right]\right]^{\frac{1}{2}} \\ &\quad F\left(\phi, \frac{\sqrt{1-v^2}}{\sin\phi}\right) \end{aligned} \qquad (75)$$

with

$$u = \frac{c}{a}, \quad v = \frac{b}{a}, \quad 1 \geq v \geq u \qquad (76)$$

and

$$\phi = \cos^{-1} u, \quad \kappa^2 = \frac{v^2 - u^2}{v^2(1-u^2)} \qquad (77)$$

The special functions appearing in Eq. (75) are the incomplete elliptic integrals of the first and second kinds respectively. They can be evaluated accurately and efficiently by means of gaussian quadrature, or by means of the Landen's ascending or ascending transformations [44].

Special Cases

Elliptical and Circular Disks: $a \geq b, c = 0; 1 \geq v, u = 0$

$$R^*_{\sqrt{A}} = \frac{1}{\sqrt{8\pi}} \sqrt{v} K(\sqrt{1-v^2}) \tag{78}$$

where $K(\cdot)$ is the complete elliptic integral of the first kind of modulus $(\sqrt{1-v^2})$.

When the elliptical disk is narrow, $v \to 0$, $K(\sqrt{1-v^2}) \to \ln(4/v)$ and therefore,

$$R^*_{\sqrt{A}} = \frac{1}{\sqrt{8\pi}} \sqrt{v} \ln\left(\frac{4}{v}\right) \tag{79}$$

For the circular disk, $v = 1$, and the dimensionless resistance, $R^*_{\sqrt{A}} = 0.3133$ and the Nusselt number, $Nu_{\sqrt{A}} = 3.1915$.

Oblate Spheroids: $a = b \geq c; v = 1, 0 \leq u \leq 1$

The dimensionless resistance for oblate spheroids is

$$R^*_{\sqrt{A}} = \frac{1}{\sqrt{8\pi}} \frac{\phi}{\sin\phi} \left[1 + \frac{\cos^2\phi}{\sin\phi} \frac{1}{2} \ln \frac{1+\sin\phi}{1-\sin\phi}\right]^{\frac{1}{2}} \tag{80}$$

with

$$\phi = \cos^{-1} u, \qquad \sin\phi = \sqrt{1-u^2} \tag{81}$$

The general expression for the oblate spheroids reduces to that for the circular disk when $u = 0$ where $Nu_{\sqrt{A}} = 3.1915$ and the sphere when $u = 1$ where $Nu_{\sqrt{A}} = 3.5449$.

Prolate Spheroids: $a \geq b = c; v = u \neq 1$

The dimensionless resistance for prolate spheroids is

$$R^*_{\sqrt{A}} = \frac{1}{\sqrt{8\pi} \sin\phi} \frac{1}{2} \ln \frac{1+\sin\phi}{1-\sin\phi} \left[\cos^2\phi + \phi \frac{\cos\phi}{\sin\phi}\right]^{\frac{1}{2}} \tag{82}$$

## Effect of Characteristic Length on Nusselt Number

The choice of characteristic length in the definition of the Nusselt number can have a significant effect on its dependence upon body shape and its aspect ratio. This will be demonstrated by considering the oblate and prolate spheroids for a range of body aspect ratios such that the Nusselt number for circular disks ($AR = 0$), spheres ($AR = 1$) and long prolate spheroids ($AR >> 1$) are included. The characteristic lengths will be the new length, $\mathcal{L} = \sqrt{A}$, and the conventional one, $\mathcal{L} = 2b$, where $2b$ is the major axis for both oblate and prolate spheroids. When the oblate and prolate spheroids reduce to the sphere, the characteristic length is the sphere diameter.

The expressions for the Nusselt numbers for these characteristic lengths are:

Oblate Spheroids

$$Nu_{2b} = 4e \left\{ \left[1 + \frac{u^2}{e} \frac{1}{2} \ln\left(\frac{1+e}{1-e}\right)\right] \cos^{-1} u \right\}^{-1} \tag{83}$$

and

$$Nu_{\sqrt{A}} = \sqrt{\frac{\pi}{2}}\, 4e \left\{ \left[1 + \frac{u^2}{e} \frac{1}{2} \ln\left(\frac{1+e}{1-e}\right)\right]^{\frac{1}{2}} \cos^{-1} u \right\}^{-1} \tag{84}$$

Prolate Spheroids

$$Nu_{2b} = 4e \left\{ \left[u^2 + \frac{u}{e} \sin^{-1} e\right] \frac{1}{2} \ln\left(\frac{1+e}{1-e}\right) \right\}^{-1} \tag{85}$$

and

$$Nu_{\sqrt{A}} = \sqrt{\frac{\pi}{2}}\, 4e \left\{ \left[u^2 + \frac{u}{e} \sin^{-1} e\right]^{\frac{1}{2}} \frac{1}{2} \ln\left(\frac{1+e}{1-e}\right) \right\}^{-1} \tag{86}$$

where $u = c/b$ and $e = \sqrt{1-u^2}$, which is called the eccentricity. The aspect ratio is $AR = u$ for the oblate spheroids and $AR = 1/u$ for the prolate spheroids.

Nusselt numbers based on $\mathcal{L} = \sqrt{A}$ and $\mathcal{L} = 2b$ are given in Tables 1 and 2, and are plotted in Fig. 2 for the oblate and prolate spheroids for a wide range of the parameter $u$. In Table 1 and Fig. 2 it is seen that $Nu_{\sqrt{A}}$ for both spheroids approach asymptotically the value for the sphere which is $Nu_{\sqrt{A}} = 2\sqrt{\pi}$; but the difference from the sphere value is very small. The circular disk ($AR = 0$) is only 10% below the sphere value, while the prolate ($AR = 20$) is 36.6% above the sphere value. The circular disk and the long prolate differ by approximately 52%; a remarkably small difference considering the great difference in the body shapes and aspect ratios. The percent difference between values of the Nusselt numbers for the two body shapes is less than 1% when the ratio of their aspect ratios is 4 and reaches approximately 10% when the ratio is 25; this is clearly seen in Fig. 2.

On the other hand, the Nusselt numbers based on $\mathcal{L} = 2b$ for the two body shapes differ considerably as seen in Table 2 and Fig. 2. Both oblate and prolate values approach the sphere value of 2; but the oblate values do not vary monotonically, and the values of the circular disk are greater than those of the sphere. There is no physical reason for the Nusselt number of the oblate spheroid of aspect ratio $AR = 0.15$ to have the optimum value. The difference between the values for the prolate of aspect ratio $AR = 20$ and the sphere is approximately 290%. This very large percent difference is to be compared with the 36.6% difference when $\mathcal{L} = \sqrt{A}$ is used.

## Effect of Body Shape and Body Aspect Ratio on Nusselt Number

The effect of the body aspect ratio on the Nusselt number can be demonstrated by examining the results for the prolate spheroid and the right circular cylinder, which are axisymmetric bodies, and the elliptical disk, which is a planar body. These bodies are shown in Fig. 3 with their maximum and minimum body lengths, $L$ and $D$, respectively. The aspect ratio is defined as $AR = L/D$.

Table 1: Comparison of $Nu_{\sqrt{A}}$ for Oblate and Prolate Spheroids vs $u$

| $u$ | Oblate Spheroid $Nu_{\sqrt{A}}$ | Prolate Spheroid $Nu_{\sqrt{A}}$ |
|---|---|---|
| 0.0000 | 3.1915 | — |
| 0.0500 | 3.2773 | 4.8412 |
| 0.1000 | 3.3419 | 4.1951 |
| 0.1500 | 3.3916 | 3.9312 |
| 0.2000 | 3.4299 | 3.7905 |
| 0.2500 | 3.4594 | 3.7064 |
| 0.3000 | 3.4819 | 3.6528 |
| 0.3500 | 3.4991 | 3.6174 |
| 0.4000 | 3.5121 | 3.5936 |
| 0.4500 | 3.5218 | 3.5773 |
| 0.5000 | 3.5290 | 3.5661 |
| 0.5500 | 3.5343 | 3.5585 |
| 0.6000 | 3.5380 | 3.5534 |
| 0.6500 | 3.5407 | 3.5500 |
| 0.7000 | 3.5424 | 3.5478 |
| 0.7500 | 3.5436 | 3.5464 |
| 0.8000 | 3.5443 | 3.5456 |
| 0.8500 | 3.5447 | 3.5452 |
| 0.9000 | 3.5448 | 3.5450 |
| 0.9500 | 3.5449 | 3.5449 |
| 1.0000 | 3.5449 | 3.5449 |

Table 2: Comparison of $Nu_{2b}$ for Oblate and Prolate Spheroids vs $u$

| $u$ | Oblate Spheroid $Nu_{2b}$ | Prolate Spheroid $Nu_{2b}$ |
|---|---|---|
| 0.0000 | 2.5464 | — |
| 0.0500 | 2.6029 | 13.7748 |
| 0.1000 | 2.6273 | 8.4260 |
| 0.1500 | 2.6299 | 6.4298 |
| 0.2000 | 2.6170 | 5.3503 |
| 0.2500 | 2.5929 | 4.6594 |
| 0.3000 | 2.5610 | 4.1714 |
| 0.3500 | 2.5238 | 3.8037 |
| 0.4000 | 2.4832 | 3.5135 |
| 0.4500 | 2.4405 | 3.2766 |
| 0.5000 | 2.3968 | 3.0779 |
| 0.5500 | 2.3529 | 2.9078 |
| 0.6000 | 2.3093 | 2.7597 |
| 0.6500 | 2.2664 | 2.6289 |
| 0.7000 | 2.2245 | 2.5122 |
| 0.7500 | 2.1837 | 2.4070 |
| 0.8000 | 2.1442 | 2.3114 |
| 0.8500 | 2.1061 | 2.2239 |
| 0.9000 | 2.0694 | 2.1435 |
| 0.9500 | 2.0340 | 2.0691 |
| 1.0000 | 2.0000 | 2.0000 |

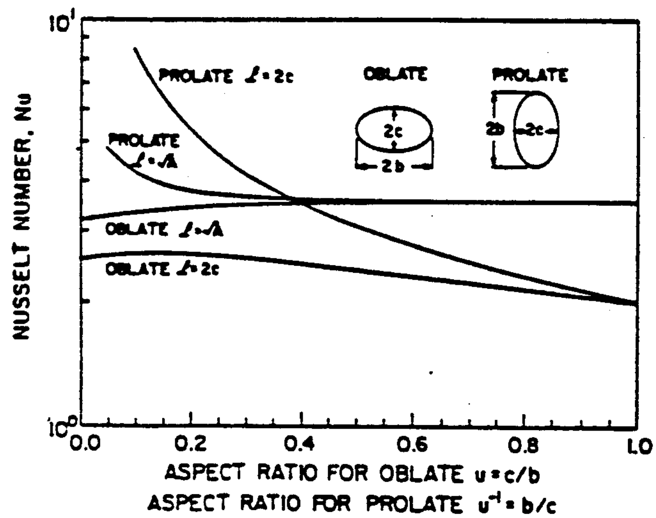



Fig. 2 Effect of body aspect ratio and characteristic length on Nusselt number for oblate and prolate spheroids

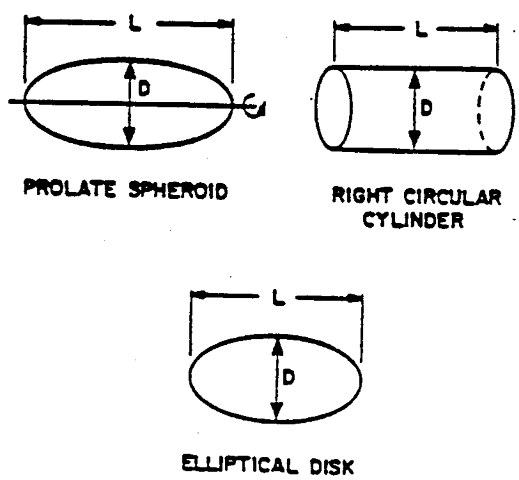



Fig. 3 Axisymmetric and planar body shapes having identical aspect ratios

The Nusselt number for the prolate spheroid and the elliptical disk can be determined from Eqs. (86) and (78) respectively. The Nusselt number for the right circular cylinder can be obtained from Smythe's correlation of his capacitance solution [39,40]:

$$Nu_{\sqrt{A}} = \frac{8 + 6.95(L/D)^{0.76}}{\sqrt{2\pi + 4\pi(L/D)}} \qquad 0 \le L/D \le 8 \qquad (87)$$

with a maximum error of 0.2%. The effect of the aspect ratio is shown in Table 3.

Table 3: Effect of Body Aspect Ratio for Prolate Spheroids, Circular Cylinders and Elliptical Disks

| AR L/D | Prolate Spheroids | Right Cylinders | Elliptical Disks |
|---|---|---|---|
| 1 | 3.545 | 3.443 | 3.192 |
| 2 | 3.566 | 3.527 | 3.288 |
| 3 | 3.628 | 3.622 | 3.434 |
| 4 | 3.706 | 3.714 | 3.579 |
| 5 | 3.790 | 3.803 | 3.716 |
| 6 | 3.875 | 3.887 | 3.845 |
| 7 | 3.959 | 3.965 | 3.952 |
| 8 | 4.040 | 4.040 | 4.080 |

The largest difference in the Nusselt number occurs when the aspect ratio is unity, and the bodies are spheres, short cylinders and circular disks. The circular disk has the smallest Nusselt number and the sphere has the largest Nusselt number, where the difference is approximately 10%. As the aspect ratio increases from 1 to 8 the Nusselt number increases slowly to a value of 4.04 for these body shapes. This is approximately 14% above the sphere value.

When the diameter of the body is used in the definition of the Nusselt number, the values reported for $AR = 1$ become 2, 1.5865, and 2.5468 for the sphere, cylinder and circular disk, respectively. We first observe that the circular disk appears to be more *conductive* than the sphere, and second, that the percent difference between the cylinder value and the circular disk is now approximately 61%.

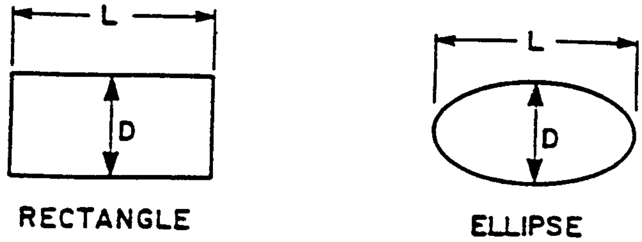
The Nusselt number for the cube can also be determined from the capacitance reported by Greenspan [36], and its value is $Nu_{\sqrt{A}} = 3.388$ and $Nu_S = 1.3831$ when its side is used as the characteristic length. When the unity aspect ratio bodies are compared, the largest percent difference is approximately 10% when $\mathcal{L} = \sqrt{A}$, and approximately 84% when $\mathcal{L} = D$ or $\mathcal{L} = S$.

When $AR \geq 5$, the prolate spheroidal Nusselt number can be approximated accurately by

$$Nu_{\sqrt{A}} = \frac{4\sqrt{8\pi(L/D)}}{\ln(4L/D)} \tag{88}$$

with an error $\leq 0.68\%$; and, as a result of the observations made above, this expression may be used to approximate
with an error $\leq 0.68\%$; and, as a result of the observations made above, this expression may be used to approximate
the Nusselt number for right circular cylinders and similar bodies as well as elliptical disks and similar bodies such as rectangular or diamond- shaped plates.

The $Nu_{\sqrt{A}}$ for the rectangular and elliptical plates are given in Fig. 4 where it can be seen that the effect of shape is very small, and the effect of aspect ratio is similar to that of the prolate spheroid.




| L/D | $Nu_{\sqrt{A}}$ (1) | $Nu_{\sqrt{A}}$ (2) |
|---|---|---|
| 1 | 3.205 | 3.192 |
| 2 | 3.303 | 3.288 |
| 3 | 3.438 | 3.434 |
| 4 | 3.553 | 3.579 |

(1) NUMERICAL RESULTS
(2) ANALYTICAL RESULTS

Fig. 4 Comparison of Nusselt numbers for rectangular and elliptical plates

The Smythe right-circular cylinder correlation can be compared with the oblate spheroidal solution when the body aspect ratio is less than unity. The Nusselt number for the oblate spheroid is greater than that of the cylinder for all $0 < AR \leq 1$, and the maximum difference of approximately 3.3% occurs at $AR \approx 0.6$.

## Nusselt Number for Disks and Plates

The Nusselt number for disks of arbitrary shape can be obtained from the reported data on the capacitance of disks [45] by the use of Eqs. (42) and (43). With the exception of the circular disk, the data for all shapes were obtained from numerical methods such as the method of moments [45]. The results are given in Fig. 5 for a range of body shapes. The body aspect ratio defined as the ratio of the minimum to maximum body lengths is unity for the circle and the square; and two for the semi-circle and the 2 : 1 diamond. The aspect ratios of the hexagon, 90° sector and the 2 : 1 right triangle are difficult to define. The very close agreement between the analytical result for the circle and the numerical result for the square is remarkable, because the difference is less than 0.5%; and even the hexagon result is within 0.4% of the circle result. The semi-circle and 2 : 1 diamond results also differ by less than 0.5%, and their Nusselt numbers are very close to those of rectangular and elliptical plates having 2 : 1 aspect ratios (See Fig. 4). The 2 : 1 right triangle and the 1.65 : 1 diamond have Nusselt numbers which are within 1.0% of each other and only 8.1% above the circle result.

From an examination of the Nusselt numbers in Figs. 4 and 5, it can be concluded that body shapes having identical or similar aspect ratios will have similar Nusselt numbers.

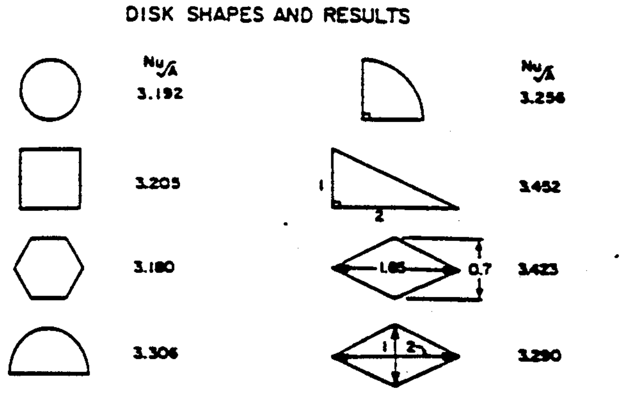



Fig. 5 Nusselt Numbers for disks of arbitrary shape and aspect ratio

## Summary and Conclusions

Relationships between the dimensionless diffusive resistance, shape factor, capacitance, and the Nusselt and Sherwood numbers have been developed. A review of the numerous characteristic lengths used in heat and mass transfer, and some conclusions regarding the appropriate length for natural convection from isothermal bodies of complex shape have been presented. An analytical solution for conduction from isothermal ellipsoids into an *infinite* domain is developed, and special cases such as oblate and prolate spheroids, spheres, elliptical and circular disks are given.

The solutions for the oblate and prolate spheroids were used to demonstrate the effect of the selection of the characteristic body length on the Nusselt number. It was shown that the square root of the body surface area is far superior to the *classical* body lengths proposed by other authors. When the appropriate body length is used to define the Nusselt number or the other dimensionless groups, the results for a wide range of body shapes and aspect ratios lie in a relatively narrow range for *similar* body shapes having *similar* aspect ratios as shown in Fig. 6.

Although the solutions presented here are for convex bodies of arbitrary shape and aspect ratio, they can be used to approximate the Nusselt number for non-convex bodies such as the bi-sphere which consists of two identical spheres in point contact. Considering this body to have an aspect ratio of two and using the result for a prolate spheroid of the same aspect ratio, one would predict a Nusselt number of 3.57. The actual Nusselt number based on the value given in Ref. 35 is 3.48, which is only different by 2.6%.

Thus, it can be concluded, that the general solution for the ellipsoid with the characteristic body length based on the square root of the total body surface area, can be used to approximate the Nusselt and Sherwood numbers

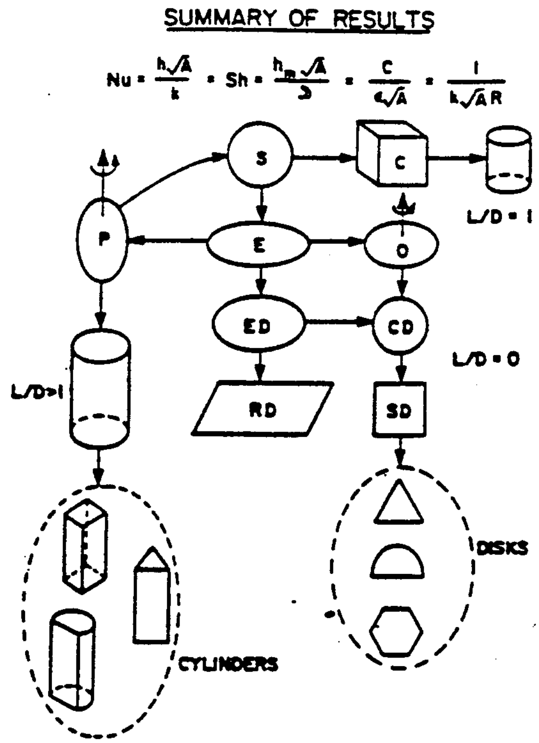



Fig. 6 Summary of results for a wide range of body shapes and aspect ratios

for bodies of complex shape with the same surface area and one or more similar body aspect ratios. The particular solutions such as the sphere can be used to approximate bodies of unity aspect ratio; the oblate spheroids can be used to approximate axisymmetric bodies of aspect ratio less than one, and the prolate spheroids can be used to approximate axisymmetric bodies of aspect ratios greater than one. These analytical solutions and the bodies which can be approximated are shown in Fig. 6.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