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The Effect of Coatings on the Steady-State and Short Time Constriction Resistance for an Arbitrary Axisymmetric Flux

The effect of a coating upon the short-time and steady-state constriction resistance is analyzed for an arbitrary axisymmetric contact spot flux. At very short times the expression obtained for R is identical to the expression for one-dimensional transient heat flow through a two-layer wall. At steady-state, the results of the analysis predict that the effect of the coating are mainly dependent on the relative thermal properties of the coating and substrate. The limiting cases, where the coating thickness approaches either zero or infinity, are discussed.

1 Introduction

When two apparently flat surfaces are pressed into contact it is well known that the actual contacting area is very small. The contacting asperities are widely separated, and if heat is to be passed through the interface there is a thermal constriction. In most cases of practical interest the contacting spots are sufficiently far apart to treat them individually as being an isolated spot on a half-plane. The constriction resistance of a spot of radius a of a half-plane with thermal conductivity k is equal to about $1/(4ka)$.

Often surfaces are coated with material having different mechanical and thermal properties from the substrate. These surface layers although apparently thin may cause large changes in the constriction resistance. In many present-day applications, such as laser annealing and microelectronics, it is important to know quantitatively the effect of a surface coating upon the constriction resistance. Recently, the effect of the coating upon the steady-state constriction resistance, for a particular specified flux, has been analyzed [1]. Here, a more general analysis is undertaken. The effect of the coating upon the short-time and also the steady-state constriction resistance for any axisymmetric flux is analyzed.

Figure 1 shows a half-space of substrate material 2 with a coating of material 1 having thickness δ. There is an axisymmetric flux $f(r)$, prescribed over a circular spot of radius a and outside this spot an adiabatic boundary condition is imposed. Initially the temperatures in materials 1 and 2 are assumed to be zero. The aim of the analysis performed in this paper is to find the effect of the coating upon the short time and steady-state constriction resistance. (On the surface $z=0$, the average temperature of the circular spot, $0<r<a$, is equal to $R\times Q$). The mathematical details are discussed later.

The results of the analysis indicate that when $\mathrm{Fo}_1<1/10$ the constriction resistance is controlled by one-dimensional heat flow. The expression obtained for R is identical to that obtained from the analysis of transient heat flow through a two-layer wall. At steady-state

$$R=R_{1c}-\frac{1}{2\pi a k_1}\left[\ln(1-K)\,\frac{a}{\delta}+C_3\left(\frac{a}{\delta}\right)^3+C_5\left(\frac{a}{\delta}\right)^5\ \ldots\right]$$

The coefficients C_3, C_5, . . . depend on $f(r)$. If the coating is thick enough so that the terms involving $(a/\delta)^3$, $(a/\delta)^5$, . . . can be neglected, or if the coating is highly conductive relative to the substrate so that $\ln(1-K)$ is large compared with C_3, C_5 . . ., then

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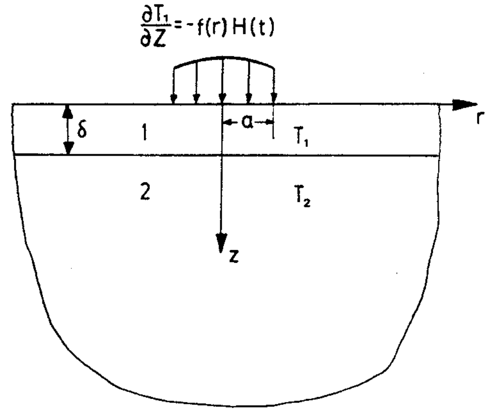

Fig. 1 Half-space with coating of thickness δ and axisymmetric flux $0<r<a$. Coating is material 1 and substrate is material 2.

$$R\approx R_{1c}-\frac{1}{2\pi a k_1}\ln(1-K)\left(\frac{a}{\delta}\right)$$

If the coating is very thin, so that $2/1-K\ \delta/a$ is somewhat less than one, then

$$R=R_{2c}-\frac{1}{\pi a k_1}\,\frac{2K}{1-K}\,\frac{2}{1-K}\,\frac{\delta}{a}[1+0(1)]$$

The higher-order terms of the thin film expansion cannot be found unless $f(r)$ is specified.

2 Problem and Analysis

A half-plane with a coating is to be analyzed to find the effect of the coating upon the constriction resistance. The transient axisymmetric problem is stated below. Referring to Fig. 1, the temperatures $T_1(r, z, t)$ and $T_2(r, z, t)$ must satisfy the heat conduction equation

$$\frac{\partial T_1}{\partial t}=\alpha_1\nabla^2T_1\quad r>0, 0<z<\delta, t>0 \tag{1}$$

$$\frac{\partial T_2}{\partial t}=\alpha_2\nabla^2T_2\quad r>0, z>\delta, t>0 \tag{2}$$

where

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}$$

Initially at $t=0$

$$T_1(r,z,0)=0\quad r>0, 0<z<\delta \tag{3}$$

$$T_2(r,z,0)=0\quad r>0, z>\delta \tag{4}$$

At the interface, $z=\delta$ there is continuity of both temperature and flux

$$T_1(r,\delta,t)=T_2(r,\delta,t) \tag{5a}$$

$$k_1\frac{\partial}{\partial z}T_1(r,\delta,t)=k_2\frac{\partial}{\partial z}T_2(r,\delta,t) \tag{5b}$$

At $z=0$, an axisymmetric heat flow is prescribed

$$\left.\begin{aligned}\frac{\partial T_1}{\partial z}(r,0,t)&=-f(r)H(t) \quad r<a\\ &=0 \quad r>a\end{aligned}\right\} \tag{6a}$$

where $H(t)$ represents a Heaviside unit function and $f(r)$ is arbitrary except that

$$k_1\int_{r=0}^{a}2\pi rf(r)dr=Q \quad (t>0) \tag{6b}$$

At this point the integral I_1 is defined as

$$I_1=\int_{x=0}^{1}f(ax)xdx=\frac{Q}{2\pi a^2k_1} \tag{6c}$$

where the integral I_1 follows from (6b) by a simple change of variables.

And finally, as $r^2+z^2\rightarrow\infty$

$$T_1(r,z,t)=T_2(r,z,t)=0 \tag{7}$$

A double transformation technique is used. A Laplace transformation and then a Hankel transformation are applied. Using equations (3) and (4), equations (1) and (2) are transformed to

$$\alpha_1\left(\frac{d^2}{dz^2}-\lambda^2\right)w_1=sw_1 \tag{8}$$

$$\alpha_2\left(\frac{d^2}{dz^2}-\lambda^2\right)w_2=sw_2 \tag{9}$$

The solutions to (8) and (9) are

$$w_1=Ae^{\eta_1 z}+Be^{-\eta_1 z} \quad \delta>z>0 \tag{10}$$

$$w_2=Ce^{-\eta_2 z} \quad z>\delta \tag{11}$$

The coefficients A, B, and C are found from the double transforms of equations (5a), (5b), and (6a). After some algebraic manipulation

$$A=\frac{F(\lambda)}{s\eta_1}\frac{\phi}{1-\phi} \tag{12}$$

$$B=\frac{F(\lambda)}{s\eta_1}\frac{1}{1-\phi} \tag{13}$$

$$C=\frac{F(\lambda)}{s\eta_1}\frac{2k_1\eta_1}{k_1\eta_1+k_2\eta_2}\exp(-\delta(\eta_1-\eta_2))\frac{1}{1-\phi} \tag{14}$$

On the plane $z=0$

$$\begin{aligned}w_1&=A+B\\&=\frac{F(\lambda)}{s\eta_1}\frac{1+\phi}{1-\phi}\end{aligned} \tag{15}$$

and since $|\phi|<1$ this can be expanded as

$$w_1=\frac{F(\lambda)}{s\eta_1}\left[1+2\sum_{i+1}^{\infty}\phi^i\right] \tag{16}$$

3 Laplace Transformation of Constriction Resistance

On the plane $z=0$

$$u_1=\int_{\lambda=0}^{\infty}w_1\lambda J_0(\lambda r)d\lambda$$

and using the formula for w_1 in (15)

$$u_1=\int_{\lambda=0}^{\infty}\frac{F(\lambda)}{s\eta_1}\frac{1+\phi}{1-\phi}\lambda J_0(\lambda r)d\lambda \tag{17}$$

The Laplace transformation of the constriction resistance is

$$\begin{aligned}\mathcal{L}[R]&=\frac{1}{\pi a^2}\int_{r=0}^{a}2\pi ru_1dr/Q\\&=\frac{2}{Qa}\int_{\lambda=0}^{\infty}\frac{F(\lambda)}{s\eta_1}\frac{1+\phi}{1-\phi}J_1(\lambda a)d\lambda\end{aligned} \tag{18}$$

If the following substitutions are made

$$\left.\begin{aligned}&\lambda=\xi/a\\ \text{and}\quad &\mathcal{H}_0[f(r);\xi/a]=a^2\mathcal{H}_0[f(ar);\xi]\end{aligned}\right\} \tag{19}$$

then

$$\mathcal{L}[R]=2/Q\int_{\xi=0}^{\infty}\frac{G(\xi)}{s\eta_1}\frac{1+\phi}{1-\phi}J_1(\xi)d\xi \tag{20}$$

where

$$G(\xi)=\int_{x=0}^{1}f(ax)xJ_0(\xi x)dx \tag{21}$$

Expressions for R which are valid for steady-state ($t=\infty$)

Nomenclature

a = contact spot radius

$C_1 = \ln\left(\frac{2k_2}{k_1+k_2}\right)=\ln(1-K)$

$C_3 = \frac{1}{16}\left(1+2\frac{I_3}{I_1}\right)\sum_{i=1}^{\infty}\frac{K^i}{i^3}$

$C_5 = -\frac{1}{128}\left(1+6\frac{I_3}{I_1}+3\frac{I_5}{I_1}\right)\sum_{i=1}^{\infty}\frac{K^i}{i^5}$

$E(x)$ = complete elliptic integral (see equation (24b))

$f(r)$ = radial distribution of axial flux

$F(\lambda)$ = $\mathcal{H}_0\{f(r);\lambda\}$

Fo_1, Fo_2 = Fourier numbers $\alpha_1 t/a^2$, $\alpha_2 t/a^2$

$G(\xi)$ = $\mathcal{H}_0\{f(ar);\xi\}$ (see equation (21))

$H(t)$ = Heavy side function

$\mathcal{H}_0\{\psi(x);\lambda\} = \int_{x=0}^{\infty}\psi(x)xJ_0(\lambda x)dx$ (Hankel transform of zeroth order)

$I_n = \int_{x=0}^{1}f(ax)x^n dx$ (see equation (26))

$J_0(\ ),J_1(\ )$ = Bessel functions

K = $(k_1-k_2)/(k_1+k_2)$

k_1,k_2 = thermal conductivity of materials 1 and 2

$\mathcal{L}\{\psi(x)\} \equiv \int_{x=0}^{\infty}\psi(x)e^{-xs}dx$

Q = flux passing through contact spot (see equation (6b))

R = constriction resistance of coated material

R_{1c} = constriction resistance of half space of material 1

R_{2c} = constriction resistance of half space of material 2 and for very short times ($\alpha_1 t/a^2 < 0.1$) are found by considering the behavior of $\mathcal{L}[R]$ as $s \to 0$ and $s \to \infty$.

4 Steady-State Constriction Resistance

In the theory of Laplace transformations it is well known that the limits $s\mathcal{L}[R]$ as $s \to 0$ will yield the steady-state solution, see, for example, Doetsch [2]. Thus from equation (20)

$$R = \frac{2a}{Q}\int_{\xi=0}^{\infty} \frac{1+\phi}{1-\phi}\,\frac{J_1(\xi)}{\xi}\int_{x=0}^{1} xf(ax)J_0(\xi x)dxd\xi \qquad (22)$$

At steady-state $\phi = K\exp(-\sigma\xi)$ which will always be less than unity. Since $|\phi| < 1$, $(1+\phi)/(1-\phi)$ can be exapnded using the binomial theorem to obtain

$$R = R_{1c} + \frac{4a}{Q}\sum_{i=1}^{\infty} K^i \Omega(\sigma_i) \qquad (23)$$

where

$$R_{1c} = \frac{2a}{Q}\int_{\xi=0}^{\infty}\frac{J_1(\xi)}{\xi}\int_{x=0}^{1} xf(ax)J_0(\xi x)dxd\xi \qquad (24a)$$

and after integrating over $0 < \xi < \infty$

$$R_{1c} = \frac{4a}{\pi Q}\int_{x=0}^{1} f(ax)xE(x)\,dx \qquad (24b)$$

In equation (23), the term R_{1c} represents the constriction resistance of a spot on a half-space of material 1. The presence of the substrate is accounted for by the summation $i = 1, 2, \ldots \infty$.

Equation (6.626) on page 715 of [3] gives

$$\int_{\xi=0}^{\infty} d\xi e^{-\sigma\xi}J_0(\xi x)\,\frac{J_1(\xi)}{\xi}$$

$$= \frac{1}{2\sigma}\sum_{m=0}^{\infty}\frac{(2m)!2^{-2m}}{(m!)^2(m+1)}F(-m,-1-m;1;x^2)\left(-\frac{1}{\sigma^2}\right)^m$$

Using this equation and integrating over $0<x<1$, $\Omega(\sigma_i)$ is found to be

$$\Omega(\sigma_i) = I_1\left[\frac{1}{2\sigma_i} - \left(\frac{1}{8}+\frac{I_3}{4I_1}\right)\frac{1}{\sigma_i{}^3} + \left(\frac{I_5}{128I_1}+\frac{I_3}{64I_1} + \frac{1}{384}\right)\frac{24}{\sigma_i 5}\ldots\right] \qquad (25)$$

where

$$I_n \equiv \int_{x=0}^{1} x^n f(ax)\,dx \qquad (26)$$

The integral I_1 has been previously defined in equation (6c) and the only restriction on $f(ax)$ is that I_1 be constant. The integrals I_3 and I_5 will have the same units as I_1 since x is dimensionless. The integrals I_n are similar to moments of $f(ax)$ about $x=0$.

The expansion given in (25) is useful when $\sigma > 1$ and can be used in (23) to give

$$R = R_{1c} - \frac{1}{2\pi a k_1}\left[\ln(1-K)\left(\frac{a}{\delta}\right) + C_3\left(\frac{a}{\delta}\right)^3 + C_5\left(\frac{a}{\delta}\right)^5\ldots\right] \qquad (27)$$

This expression shows the effect of the coating upon the constriction resistance. It is useful for small values of a/δ that correspond to thick coatings. In the next section an asymptotic form that shows the thin film behavior is found.

Effect of a Thin Coating – $\delta/a \to 0$. After integrating by parts equation (22) may be reexpressed as

$$R = R_{2c} - \frac{2a}{Q}\,\frac{2K}{1-K}\psi(\sigma) \qquad (28a)$$

and

$$\psi(\sigma) = \int_{x=0}^{1} dxf(ax)x\int_{\xi=0}^{\infty} d\xi\,\frac{J_1(\xi)}{\xi}J_0(\xi x)\left[\frac{1-e^{-\sigma\xi}}{1-Ke^{-\sigma\xi}}\right] \qquad (28b)$$

where $R_{2c} = R_{1c}\ k_1/k_2$. In this section the behavior of the integral given in (28b) is considered as $\sigma \to 0$.

Within the interval $0 < \xi < 1 - K/\sigma$

$$\left[\frac{1-e^{-\sigma\xi}}{1-Ke^{-\sigma\xi}}\right] = \left(\frac{\sigma\xi}{1-K}\right) + 0\left(\left(\frac{\sigma\xi}{1-K}\right)^2\right)$$

and results I, II, and III are obtained.

$$\text{I}\qquad \int_{x=0}^{1} dxf(ax)x\int_{\xi=0}^{\frac{1-K}{\sigma}} d\xi\,\frac{J_1(\xi)}{\xi}J_0(\xi x)\left(\frac{\sigma\xi}{1-K}\right)$$

$$= \frac{\sigma}{1-K}I_1 + 0\left(\frac{\sigma}{1-K}\right)$$

(See integral #3 on page 667 of [3].)

$$\text{II}\qquad \int_{x=0}^{1} dxf(ax)x\int_{\xi=0}^{\left(\frac{1-K}{\sigma}\right)^p} d\xi\,\frac{J_1(\xi)}{\xi}J_0(\xi x)0\left(\left(\frac{\sigma\xi}{1-K}\right)^2\right)$$

$$= 0\left(\left(\frac{\sigma}{1-K}\right)^{2-2p}\right)(0<p<1/2)$$

Nomenclature (cont.)

r = radial coordinate
s = transformed time variable
T_1, T_2 = temperature in materials 1 and 2
$\bar{T}$ = average temperature of contact spot
t = time
u_1, u_2 = Laplace transforms of T_1 and T_2
v_1, v_2 = Hankel transforms of T_1 and T_2
w_1, w_2 = double transforms of T_1 and T_2
W = $(\delta/a)/\sqrt{\mathrm{Fo}_1}$
z = axial coordinate

Greek Symbols

α_1, α_2 = thermal diffusivities of materials 1 and 2
γ = $(k_1/\sqrt{\alpha_1} - k_2/\sqrt{\alpha_2})/(k_1/\sqrt{\alpha_1} + k_2/\sqrt{\alpha_2})$
δ = coating thickness
λ = ξ/a
η_1 = $\sqrt{\lambda^2 + s/\alpha_1} = \sqrt{(\xi/a)^2 + s/\alpha_1}$
η_2 = $\sqrt{\lambda^2 + s/\alpha_2} = \sqrt{(\xi/a)^2 + s/\alpha_2}$
$\psi(\sigma)$ = $\int_{x=0}^{1} dxf(ax)x\int_{\xi=0}^{\infty} d\xi\,\frac{J_1(\xi)}{\xi}J_0(\xi x)\left[\frac{1-e^{-\sigma\xi}}{1-Ke^{-\sigma\xi}}\right]$
ϕ = $\frac{k_1\eta_1 - k_2\eta_2}{k_1\eta_1 + k_2\eta_2}\exp(-2\delta\eta_1)$; $|\phi| < 1$
λ = Hankel transform of radial coordinate
ρ = R/R_{2c}
σ = $2\delta/a$
σ_i = $2i\delta/a$
$\Omega(\sigma)$ = $\int_{x=0}^{1} dxf(ax)x\int_{\xi=0}^{\infty} d\xi e^{-\sigma\xi}\frac{J_1(\xi)}{\xi}J_0(\xi x)$
θ = $\frac{k_2}{k_1}\sqrt{\frac{\alpha_1}{\alpha_2}}$

III $$\int_{x=0}^{1} dx f(ax) x \int_{\xi=\left(\frac{1-K}{\sigma}\right)^{p}}^{\frac{1-K}{\sigma}} d\xi \frac{J_1(\xi)}{\xi} J_0(\xi x) 0$$

$$\left(\left(\frac{\sigma\xi}{1-K}\right)^2\right) = 0\left(\frac{\sigma}{1-K}\right) \xi = \left(\frac{1-K}{\sigma}\right)^{p}$$

See Lemma 1 on page 304 of Sneddon [4].

For the interval $1-K/\sigma<\xi<\infty$

IV $$\int_{x=0}^{1} dx f(ax) x \int_{\xi=\frac{1-K}{\sigma}}^{\infty} d\xi \frac{J_1(\xi)}{\xi} J_0(\xi x) \left[\frac{1-e^{-\sigma\xi}}{1-Ke^{-\sigma\xi}}\right]$$

$$= 0\left(\frac{\sigma}{1-K}\right)$$

See Lemma 1 on page 304 of Sneddon [4].

Using the results I, II, III, and IV the asymptotic behavior of $\psi(\sigma)$ is

$$\psi(\sigma) = \frac{\sigma}{1-K} I_1 + 0\left(\frac{\sigma}{1-K}\right)$$

The only restrictions of $f(ax)$ are that it is piecewise continuous and that the integrated flux is Q. The 0 symbol gives no information about the next series term, which will depend on $f(r)$. At this point all that can be stated is that

$$R \sim R_{2c} - \frac{1}{\pi a k_1} \frac{2K}{(1-K)} \frac{\delta}{a} \frac{2}{1-K} [1+0(1)] \text{ as } \frac{2}{1-K}\frac{\delta}{a} \to 0 \tag{29}$$

Equation (29) compliments (27). In (29) if $\delta/a = 0$ then $R = R_{2c}$ as expected. For a small value of $\sigma/(1-K)$ the effect of the coating upon R will be as shown in (29).

5 Effect of Surface Coating on the Constriction Resistance at Short Times

By considering the behavior of $\mathcal{L}[R]$ as $s\to\infty$, an expression for R is found which is valid for very short times.

If the flux over the contact spot is uniform, then it is expected that for extremely short times the average temperature of the spot would be the same as if there were a continuous planar source distributed on the surface $z = 0$. In fact, Beck [5] has shown that for uncoated surfaces this is true. It is shown below that for any axisymmetric flux $(0<r<a)$ the constriction resistance is behaving in this manner.

Equation (20) can be written as

$$\mathcal{L}[R] = \frac{2}{Q}\int_{\xi=0}^{\xi_1} \frac{G(\xi)}{s\eta_1} \frac{1+\phi}{1-\phi} J_1(\xi) d\xi + \frac{2}{Q}\int_{\xi=\xi_1}^{\infty} \frac{G(\xi)}{s\eta_1}\frac{1+\phi}{1-\phi} J_1(\xi) d\xi \tag{30}$$

We can choose ξ_1 large enough so that the second integral is negligible.

For a fixed value of ξ_1 we can take s sufficiently large so that the minimum of $(s/\alpha_1, s/\alpha_2) >> \xi_1{}^2/a^2$. Under these conditions

$$\eta_1 = \sqrt{s/\alpha_1}, \eta_2 = \sqrt{s/\alpha_2}$$

and

$$\phi = \gamma \exp[-2\delta\sqrt{s/\alpha_1}]$$

and then (30) can be written as

$$\mathcal{L}[R] = \frac{2}{Q}\frac{\sqrt{\alpha_1}}{s^{3/2}} \frac{1+\phi}{1-\phi} \int_{\xi=0}^{\xi_1} G(\xi) J_1(\xi) d\xi \tag{31}$$

For large ξ_1 the integral in (31) is equal to $Q/2\pi a^2 k_1$, so that

$$\mathcal{L}[R] = \frac{\sqrt{\alpha_1}}{k_1 \pi a^2}\frac{1}{s^{3/2}}\frac{1+\phi}{1-\phi} = \frac{\sqrt{\alpha_1}}{k_1\pi a^2 s^{3/2}}\left[1+2\sum_{i=1}^{\infty}\phi^i\right] \tag{32}$$

This can be inverted to give

$$R = \frac{\sqrt{4\alpha_1}}{k_1\pi a^2}\left\{\sqrt{\frac{t}{\pi}} + 2\sum_{i=1}^{\infty}\gamma^i\left[\sqrt{\frac{t}{\pi}}\exp\left(-\frac{i^2\delta^2}{\alpha_1 t}\right) - \left(\frac{i\delta}{\sqrt{\alpha_1}}\right)\operatorname{erfc}\left(\frac{i\delta}{\sqrt{\alpha_1 t}}\right)\right]\right\} \tag{33}$$

This is identical to the surface temperature on a two-layer wall with a prescribed flux density $\{(Q/\pi a^2)/Q\}$ being imposed on the surface, see Griffith and Horton [6].

Equation (33) can be rewritten

$$R = \frac{2}{k_1\pi a}\sqrt{\frac{\mathrm{Fo}_1}{\pi}} + \frac{4}{k_1 \pi a}\sum_{i=1}^{\infty}\gamma^i\left\{\sqrt{\frac{\mathrm{Fo}_1}{\pi}}\exp\left(-\frac{i^2(\delta/a)^2}{\mathrm{Fo}_1}\right) - i\left(\frac{\delta}{a}\right)\operatorname{erfc}\left(\frac{i(\delta/a)}{\sqrt{\mathrm{Fo}_1}}\right)\right\} \tag{34}$$

For the case of a uniform flux for $r<a$ on a half-space of material 1, Beck [5] shows that R behaves as the first term in (34) for values of $\mathrm{Fo}_1<0.1$. The effect of the substrate material 2 is represented by the summed quantity. In (34) if $\delta/a = 0$, then after some simple algebra it follows that

$$R = R_{2c} = \frac{2}{k_2\pi a}\sqrt{\frac{\mathrm{Fo}_2}{\pi}} \tag{35}$$

On the other hand, if $\{(\delta/a)/\sqrt{\mathrm{Fo}_1}\}$ is greater than about 2

Table 1 Values of R_{1c}, C_1, C_3 and C_5 in equation (27)

$f(r)$	R_{1c}	C_1	C_3 $(\Sigma(1)^i i^{-3} = 1.202)$ $(\Sigma(-1)^i i^{-3} = -0.902)$	C_5 $(\Sigma(1)^i i^{-5} = 1.037)$ $(\Sigma(-1)^i i^{-5} = -0.972)$
(*i*) Point source at center	$\frac{1}{\pi a k_1}$	$ln(1-k)$	$0.0625\sum_{i=1}^{\infty}\frac{K^i}{i^3}$	$-0.0078\sum_{i=1}^{\infty}\frac{K^i}{i^5}$
(*ii*) $\frac{Q}{k_1\pi a^2}$	$\frac{8}{3\pi^2 a k_1}$	$\ln(1-K)$	$0.1250\sum_{i=1}^{\infty}\frac{K^i}{i^3}$	$-0.0391\sum_{i=1}^{\infty}\frac{K^i}{i^5}$
(*iii*) $\frac{Q}{2k_1\pi a\sqrt{a^2-r^2}}$	$\frac{1}{4ak_1}$	$\ln(1-K)$	$0.1458\sum_{i=1}^{\infty}\frac{K^i}{i^3}$	$-0.0516\sum_{i=1}^{\infty}\frac{K^i}{i^5}$
(*iv*) Ring Source at $r=a$	$\frac{2}{\pi^2 a k_1}$	$\ln(1-K)$	$0.1878\sum_{i=1}^{\infty}\frac{K^i}{i^3}$	$-0.0781\sum_{i=1}^{\infty}\frac{K^i}{i^5}$

$$\operatorname{erfc}\left(\frac{i(\delta/a)}{\sqrt{\mathrm{Fo}_1}}\right) \sim \sqrt{\frac{\mathrm{Fo}_1}{\pi}}\frac{1}{i}\left(\frac{a}{\delta}\right)\exp\left(-\frac{i^2(\delta/a)^2}{\mathrm{Fo}_1}\right) \tag{36}$$

then

$$R = R_{1c} = \frac{2}{k_1 \pi a}\sqrt{\frac{\mathrm{Fo}_1}{\pi}} \tag{37}$$

6 Results and Discussion

6.1 Steady-State.

6.1.1 Effect of Coating. If $\delta/a = \infty$ then $R = R_{1c}$. The effect of a coating can be found using equation (27) which is

$$R = R_{1c} - \frac{1}{2\pi a k_1}\left[C_1\left(\frac{a}{\delta}\right) + C_3\left(\frac{a}{\delta}\right)^3 + C_5\left(\frac{a}{\delta}\right)^5 + \ldots\right] \tag{27}$$

The coefficients $C_3, C_5, \ldots$ depend on k_1, k_2 and $f(r)$ while the leading term C_1 is only dependent on k_1 and k_2. In Table 1 values of R_{1c}, C_1, C_3, and C_5 are given for

(*i*) $f(r)$ as a point source at the center of the contact
(*ii*) uniform flux over the contact
(*iii*) flux which causes an isothermal contact spot
(*iv*) $f(r)$ as a ring source located at the periphery $r = a$

Cases (*i*) and (*iv*) represent the two extreme cases. (The values C_3, C_5, etc., corresponding to (*i*) will be smaller than any other flux and the values C_3, C_5, etc. corresponding to (*iv*) will be larger than any other flux.)

From the values in Table 1, it is apparent that if $a/\delta < 1/2$ (say) then the effect of the coating can be predicted from the leading term of the series which is $[-\ln(1-K)/2\pi a k_1(a/\delta)]$. For thinner coatings the terms involving C_3, C_5, etc., which depend on $f(r)$, become increasingly important.

If the coating is highly conductive relative to the substrate, then $K \to 1$, and there is a logarithmic singularity that arises because of the adiabatic boundary condition at $z = \delta$. In this case, even for large values of a/δ the leading series term will be dominant.

Recently, the steady-state temperature distribution in a large circular plate heated by a disk heat source has been studied by Mehta and Bose [7]. In their analysis they specify uniform flux over $0 < r < a$ with an adiabatic condition outside the spot and at $z = \delta$ they set $T_1 = 0$. These conditions correspond to case (*ii*) with $K = -1$. Their expression for the average contact spot temperature, (their equation (13)), contains a minor error. (The 4 preceding the summation should be replaced with 2.) Their results are valid for $\sigma > 1$.

6.1.2 Effect of a Thin Coating. If $\delta/a = 0$ then $R = R_{2c}$ and for very thin coatings R can be found using equation (29) which is

$$R \sim R_{2c} - \frac{1}{\pi a k_1}\frac{4K}{(1-K)^2}\frac{\delta}{a} \quad \text{as} \quad \frac{2}{1-K}\frac{\delta}{a} \to 0 \tag{29}$$

To find the higher-order terms in the thin coating expansion, $f(r)$ must be given. For example, if $f(r) = Q/2\pi a k_1\sqrt{a^2 - r^2}$ then $\psi(\sigma)$ in (28*b*) is

$$\psi(\sigma) = \frac{Q}{2\pi a^2 k_1}\left[\frac{\sigma}{1-K} + 0\left(\left(\frac{\sigma}{1-K}\right)^{3/2}\right)\right]$$

and if $f(r) = Q/\pi a^2 k_1$ then $\psi(\sigma)$ is

$$\psi(\sigma) = \frac{Q}{\pi a^2 k_1}\left[\frac{1}{2}\frac{\sigma}{1-K} + 0\left(\left(\frac{\sigma}{1-K}\right)^2 \ln\left(\frac{\sigma}{1-K}\right)\right)\right]$$

The higher-order terms in the thin film asymptotic expansion cannot be found unless $f(r)$ is given. This is in contrast to equation (27), which gives the effect of the coating as a power series in (a/δ), the coefficients of which depend on $f(r)$.

If there is a thin layer of a poor conductor it is reasonable to expect that the heat flow through the coating will be mainly in the $z =$ direction. The extra resistance due to the coating will be a one-dimensional effect and

$$R_{1-d} = \Delta T_{1-d}/Q = \frac{1}{\pi a}\frac{\delta}{a}\left[\frac{1}{k_1} - \frac{1}{k_2}\right].$$

The total constriction resistance is

$$R = R_{2c} + R_{1-d}$$

and if $k_1 << k_2$ then

$$R = R_{2c} + \frac{1}{\pi a k_1}\frac{\delta}{a} \tag{38a}$$

This is the same result as predicted by equation (29) and is valid when

$$\frac{2}{1-K}\frac{\delta}{a} \approx \frac{\delta}{a} << 1.$$

On the other hand, when there is a thin coating of highly conductive material, then it is reasonable to assume that the temperature within the coating does not vary across its thickness δ. If $T_1(r,\delta,t) = T_2(r,\delta,t)$ and if continuity of flux is mountained, then at steady-state the appropriate boundary condition at $z = \delta \simeq 0$ becomes

$$\frac{\partial^2 T_2}{\partial z^2} - \frac{1}{\delta}\left[\frac{k_2}{k_1}\frac{\partial T_2}{\partial z} + f(r)\right] = 0 \text{ at } z = 0$$

This is discussed on pages 23–24 of Carslaw and Jaeger [8]. Using this boundary condition

$$R = R_{2c} - \frac{2a}{Q}\left(\frac{k_1}{k_2}\right)^2\frac{\delta}{a}\int_{\xi=0}^{\infty} dx f(ax)x$$

$$\int_{\xi=0}^{\infty} d\xi \frac{J_1(\xi)J_0(\xi x)}{\left(1 + \frac{k_1}{k_2}\frac{\delta}{a}\xi\right)}$$

and from an analysis similar to that in section 4.2

$$R = R_{2c} - \frac{1}{\pi a k_1}\frac{k_1}{k_2}\left[\frac{k_1}{k_2}\frac{\delta}{a} + 0\left(\frac{k_1}{k_2}\frac{\delta}{a}\right)\right] \tag{38b}$$

When $k_1 >> k_2$ equation (29) gives the same result and is valid only when $2/1-K\ \delta/a = k_1/k_2\ \delta/a << 1$. Apparently as k_1/k_2 becomes large δ/a must be very small if (29) is to be valid.

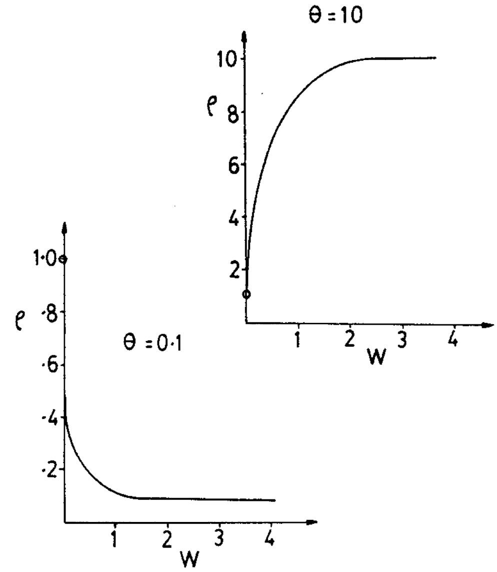

Fig. 2 Short-time values of ρ as a function of $W = (\delta/a)/\sqrt{\mathrm{Fo}_1}$ for a resistive layer $k_1/k_2 = \alpha_1/\alpha_2 = 0.01$ ($\theta = 10$) and for a conductive layer $k_1/k_2 = \alpha_1/\alpha_2 = 100.$ ($\theta = 0.1$)

In summary, the effect of a thin surface coating is given in equation (29). If the surface coating is a poor conductor of heat, then the extra resistance caused by the coating is a one-dimensional effect and is given in equation (38*a*), which is valid when δ/a is somewhat less than 1. On the other hand, if the film is highly conductive, then the flow within the surface layer is in the radial direction and the decrease in the constriction resistance is given in equation (38*b*), which is valid when δ/a is somewhat less than k_2/k_1. This shortcoming is offset by the fact that as k_2/k_1 becomes very small, euation (27) becomes useful for smaller values of δ/a.

6.2 Short Time Constriction Resistance. To investigate the effect of a coating at short time, a dimensionless constriction resistance $\rho = R/R_{2c}$ is defined. Using (34) and (35)

$$\rho = \theta\left[1 + 2\sum_{i=1}^{\infty}\left(\frac{1-\theta}{1+\theta}\right)^{i}\{\exp(-i^2W^2) - \sqrt{\pi}pW\,\mathrm{erfc}(iW)\}\right] \tag{39}$$

where

$$\theta = \frac{k_2}{k_1}\sqrt{\frac{\alpha_1}{\alpha_2}} \text{ and } W = \frac{(\delta/a)}{\sqrt{\mathrm{Fo}_1}}$$

In (39) if $W \rightarrow 0$ then $\rho \rightarrow 1$ and for large values of W (about $W > 2$) $\rho \rightarrow \theta$. In other words, for a given δ/a, at small values of Fo_1 the coating controls ρ and at large values of Fo_1 the substrate dominates.

In Fig. 2 this effect is shown for a resistive layer $\theta = 10$ and a conductive layer $\theta = 1/10$. If we assume that $k_1/k_2 \approx \alpha_1/\alpha_2$, then it can be seen that the effect of the coating on the short time values of ρ are about equal to the square root of the steady-state maximum.

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