NON-ITERATIVE CONTROL VOLUME APPROACH TO ONE-DIMENSIONAL STEADY CONDUCTION WITH CONVECTION: APPLICATIONS TO EXTENDED SURFACES

M. M. Yovanovich
Thermal Engineering Group
Department of Mechanical Engineering
University of Waterloo
Waterloo, Ontario, Canada

ABSTRACT

A numerical method based upon the integral formulation of the conduction equation has been applied to extended surfaces of arbitrary profile, with nonuniform heat transfer coefficient, end cooling and imperfect base contact. The method utilizes temperature excess coefficients which relate local and adjacent upstream temperature excesses. General expressions of the temperature excess coefficients, local temperature excess, total fin heat loss rate and fin efficiency are developed for longitudinal fins, spines and pins, and circular annular fins. Several examples discussed in Kern and Kraus are considered to demonstrate the simplicity, directness and accuracy of the method which can be easily adapted to programmable calculators or microcomputers.

NOMENCLATURE

A	conduction area
a	fin end half-thickness
Bi	Biot number
Bic	Base Biot number
Bie	End Biot number
b	fin base half-thickness
C_j	temperature excess coefficient ($1 \leq j \leq N$)
CV	control volume
h(x)	nonuniform heat transfer coefficient
h_j	mean value of h(x) over CV_j
$\bar{h}$	overall mean value of h(x)
k	thermal conductivity
L	fin length
m	fin parameter ($= \sqrt{hP/kA}$)
N	number of control volumes
P	fin perimeter
Q	heat flow rate
q	heat flux
R_c	contact resistance
r	radial coordinate
S	convection surface
T	temperature
x	cartesian coordinate
w	fin width
y	fin half-thickness at arbitrary location
α	circular fin inner radius
β	circular fin outer radius
γ	fin parameter ($= Bi(L/b)^2$)
ϵ	fin half-thickness ratio ($=a/b$)
ξ	temperature excess coefficient parameter ($\xi =$ 0 or 1)
η	fin efficiency
θ	temperature excess
λ	temperature excess coefficient parameter ($\lambda = 1$ or 2)
μ	fin profile parameter
ν	heat transfer coefficient parameter

ρ circular annular fin radii parameter ($= 1 - \frac{\alpha}{\beta}$)

SUBSCRIPTS

b base

c contact

ce convection loss from end

cj convection loss from jth control volume

f ambient condition

i ideal

j jth control volume ($1 \leq j \leq N$)

j+1,j boundary between local and upstream control volumes

j,j-1 boundary between local and downstream control volumes

s convection surface

n outward normal

SUPERSCRIPT

$-$ mean value of parameter

$\rightarrow$ vector notation

INTRODUCTION

Analytical solutions to extended surfaces (or one dimensional conduction with convection problems) are available for a relatively small number of special cases such as longitudinal fins, spines or pins and circular annular fins having uniform film coefficients, isothermal base and insulated end (1-6). The fin profiles which have been examined (6) are i) rectangular, ii) convex parabolic, iii) triangular and iv) concave parabolic. Special functions are frequently encountered in these solutions, e.g., modified Bessel functions of integer and fractional order, and Gamma functions. Many of the special functions are not tabulated, nor are they available in computer libraries, therefore extensive numerical computations are required to determine temperatures, total heat flow rates and fin efficiencies.

Analytical methods cannot be employed for fins of arbitrary profile, with variable film coefficients, imperfect base contact and end cooling. It is therefore necessary to consider numerical methods such as finite difference (1-6) or finite element (4) to obtain discrete approximate solutions.

The purpose of this paper is to develop by means of the integral form of the conduction equation or physical approach (1-4) a general simple, accurate, non-iterative numerical method which can be applied to any one-dimensional, steady-state conduction with convection problems. To demonstrate the simplicity and accuracy of the method, it will be applied to extended surfaces of arbitrary profile, nonuniform lateral heat transfer coefficient, with base contact conductance and end cooling. Several fin examples given in Kern and Kraus (6) will be discussed to demonstrate the applicability of this novel numerical method to programmable calculators or microcomputers.

Integral Form of Conduction Equation Applied to Control Volumes

The novel, non-iterative control volume or physical approach to the solution of any one-dimensional, steady-state, conduction with convection problems presented in this paper is based upon the integral form of the conduction equation

$$\iint_S \vec{q} \cdot \vec{n} \, dS = 0 \tag{1}$$

where $\vec{q}$ is the outward directed heat flux vector at the boundary of the system under consideration, and $\vec{q}.\vec{n}$ is the outward normal component of the heat flux vector q_n. In general equation (1) can be written as

$$\sum_{j=1}^{6} \bar{q}_{nj} \, S_j = 0 \tag{2}$$

where $\bar{q}_{nj}$ is the mean value of the normal component of the heat flux vector acting over the jth surface, defined as

$$\bar{q}_{nj} = \frac{1}{S_j} \iint_{S_j} q_{nj} \, dS_j \tag{3}$$

Over convection boundaries we will take

$$\bar{q}_{nj} = h_j \; \bar{\theta}_{sj} \tag{4}$$

where h_j is the mean value of the nonuniform heat transfer coefficient over the jth boundary and $\bar{\theta}_{sj}$ is some mean value of the surface temperature excess defined as $(\bar{T}_{sj} - T_f)$, where

$$\bar{\theta}_{sf} = \frac{1}{S_j} \iint_{S_j} (T_{sj} - T_f) \, dS_j \tag{5}$$

and

$$h_j = \frac{1}{S_j} \iint_{S_j} h(x) \, dS_j \tag{6}$$

At conduction boundaries

$$\bar{q}_{nj} = -k_j \left(\frac{d\bar{\theta}}{dn}\right)_j \tag{7}$$

where k_j is the local thermal conductivity and $(d\bar{\theta}/dn)_j$ is the mean temperature excess gradient normal to the jth boundary,

$$\left(\frac{d\bar{\theta}}{dn}\right)_j = \frac{1}{S_j} \iint_{S_j} \left(\frac{d\theta}{dn}\right)_j \, dS_j \tag{8}$$

Over adiabatic boundaries

$$q_n = 0 \tag{9}$$

The control volume method conserves the geometry (conduction and convection areas), the thermal conductivity k_j and approximates the mean values of the normal temperature excess gradient at conduction areas and the mean value of the temperature excess at convection areas.

Outline of the Temperature Excess Coefficient Methodology (TECM)

The non-iterative numerical method is based upon the control volume or physical approach which can be applied to any one-dimensional conduction with convection problem. It is based upon the concept of temperature excess coefficients and the following methodology is employed:

1. Subdivide the one-dimensional system (e.g. extended surfaces) into N control volumes as shown in Figure 1. For most systems control volumes of equal length yield the simplest formulations of the temperature excess coefficients.

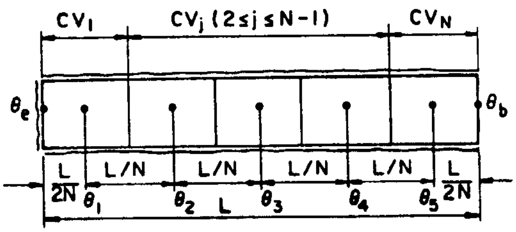

Fig. 1 Subdivision of one-dimensional system

2. Denote the end (boundary) control volume CV_1 and the base (boundary) control volume CV_N. The remaining (internal) control volumes are denoted CV_j with $2 \le j \le (N-1)$ as shown in Fig. 1.

3. Let θ_1, θ_2, θ_3, ... θ_N represent the effective temperature excess associated with each control volume. This temperature excess is assigned to the mid-point of each control volume as shown in Fig. 1. The temperature excess assigned to the fin base and end are denoted θ_b and θ_e, respectively.

4. By means of a heat balance over the end of the first control volume, CV_1, as shown in Fig. 2, derive a relationship between the end temperature excess θ_e and the control volume temperature excess θ_1, and define the end temperature excess coefficient C_e:

$$\theta_e = C_e \theta_1 \tag{10}$$

5. By means of a heat balance over the boundaries of the first control volume, Fig. 2, derive the relationship between the local temperature excess θ_1 and the upstream temperature excess θ_2,

$$\theta_1 = C_1 \theta_2 \tag{11}$$

where C_1 is the temperature excess coefficient for CV_1.

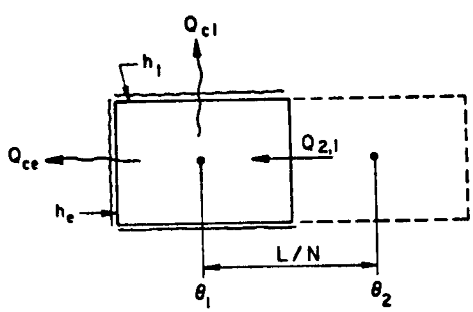

Fig. 2 Heat balance over end control volume

6. By means of a heat balance over the boundaries of an arbitrary internal control volume CV_j, Fig. 3, derive the relationship between the local temperature excess θ_j and the upstream temperature excess θ_{j+1}, using the fact that $\theta_{j-1} = C_{j-1}\,\theta_j$:

$$\theta_j = C_j \theta_{j+1} \tag{12}$$

where C_j is the temperature excess coefficient for all internal control volumes and $2 \le j \le (N-1)$.

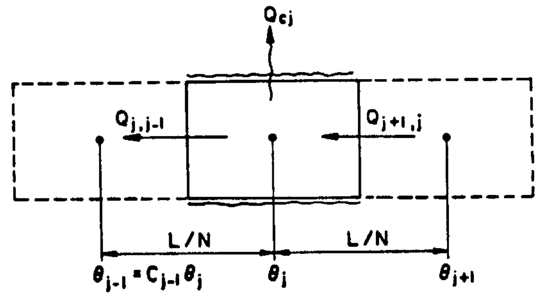

Fig. 3 Heat balance over arbitrary internal control volume

7. By means of a heat balance over the base control volume CV_N, Fig. 4, derive the relationship between the local temperature excess θ_N and the fin base temperature excess θ_b', using the fact that $\theta_{N-1} = C_{N-1}\theta_N$:

$$\theta_N = C_N \theta_b' \tag{13}$$

where C_N is the base control volume temperature excess coefficient.

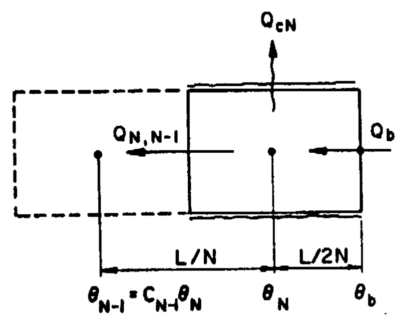

Fig. 4 Heat balance over base control volume

8. By means of a heat balance over the base conduction area, Fig. 4, derive the relationship between the fin base temperature excess θ_b' due to base contact conductance, and the base temperature excess θ_b, using the fact that $\theta_N = C_N\theta_b'$:

$$\theta_b' = C_c \theta_b \tag{14}$$

where C_c is the temperature excess coefficient due to imperfect contact at the base.

9. Compute the numerical value of the end temperature excess coefficient first; then compute directly the numerical values of all upstream temperature excess coefficients C_1, C_2,..., C_j, ...C_N, C_c.

10. Having determined the numerical values of all temperature excess coefficients, compute directly the temperature excess of each control volume beginning with the base control volume and the known temperature excess θ_b:

$$\theta_b$$

$$\theta_b' = C_c\theta_b$$

$$\theta_N = C_N\theta_b'$$

$$\theta_{N-1} = C_{N-1}\theta_N$$

$$\vdots \qquad \vdots$$

$$\theta_j = C_j\theta_{j+1}$$

$$\vdots \qquad \vdots$$

$$\theta_2 = C_2\theta_3$$

$$\theta_1 = C_1\theta_2 \tag{15}$$

$$\theta_e = C_e\theta_1$$

11. Having determined the effective temperature excess at each control volume, compute directly the convective heat loss rate from each control volume Q_{cj} where $1 \le j \le N$, and the end heat loss rate Q_{ce}. The total heat loss rate from the fin is equal to the sum of the control volume heat loss rates plus the end loss rate:

$$Q = \sum_{j=1}^{N} h_j\theta_j S_j + h_e\theta_e A_e \tag{16}$$

12. The fin efficiency η can be determined by means of the following relationship:

$$\eta = \frac{\sum_{j=1}^{N} h_j\theta_j S_j + h_e\theta_e A_e}{(\bar{h}S + h_e A_e)\,\theta_b} = \frac{Q}{Q_i} \tag{17}$$

where Q_i is the fin ideal heat loss rate.

In "Eq. (17)" h_e is the end heat transfer coefficient, and $\bar{h}S$ is given by

$$\bar{h}S = \iint_S h(x)\, dS \tag{18}$$

where $h(x)$ is the nonuniform lateral heat transfer coefficient.

13. The general procedure outlined above can be repeated for any number of control volumes beginning with $N \ge 3$, until the local temperature excess θ_j, or the total heat loss rate Q, or the fin efficiency η satisfy the specified convergence tolerances.

The methodology outlined above can be applied to extended surfaces such as a) longitudinal fins, b) spines or pins, and c) circular annular fins having the following geometric and thermal characteristics: i) arbitrary fin profile, ii) long and short fins, iii) truncated or not truncated, iv) with or without end cooling v) uniform or variable lateral film coefficients, vi) perfect or imperfect contact at the base, and vii) variable thermal conductivity. The arbitrary profiles are shown in Fig. 5 where μ is the fin profile parameter.

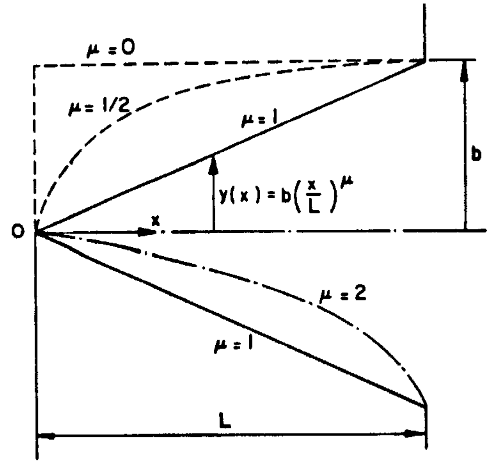

Fig. 5 Profiles of non-truncated fins with coordinate

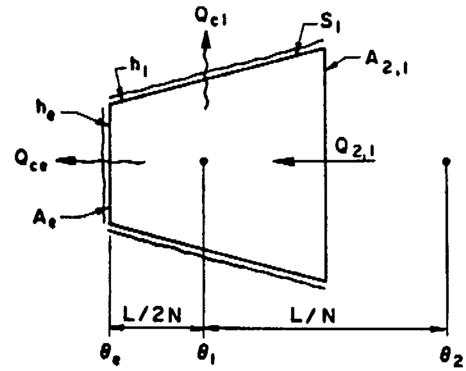

Fig. 6 Geometric and thermal relationships for end control volume

<u>Temperature Excess Coefficients C_e and C_1</u>

The general expressions of the temperature excess coefficients C_e and C_1 will be derived by means of a heat balance over the boundaries of the end control volume CV_1. Fig. 6 shows the end, local and upstream temperature excess and their locations. The conduction area between θ_1 and θ_2 is denoted $A_{2,1}$ and the conduction flow rate from CV_2 into CV_1 is denoted $Q_{2,1}$. The lateral convection surface is denoted S_1 and the convection heat loss rate from CV_1 through this surface is denoted Q_{c1}. The mean value of the film coefficient is denoted h_1; it can be determined analytically or numerically given the distribution $h(x)$ over the lateral surface. The end area is A_e, the end film coefficient is h_e and the convection heat loss rate from CV_1 through the end is Q_{ce}.

A heat balance over CV_1 gives

$$Q_{2,1} = Q_{c1} + Q_{ce} \tag{19}$$

The following exact expressions can be used to determine the conduction and convection heat flow rates:

$$Q_{2,1} = [\bar{q}A]_{2,1} = [k\frac{d\bar{\theta}}{dx}]_{2,1} \cdot A_{2,1} \tag{20}$$

$$Q_{c1} = \iint_{S_1} h(x)\theta(x)\, dS_1 \tag{21}$$

$$Q_{ce} = h_e A_e \theta_e = [k\frac{d\bar{\theta}}{dx}]_{x=o} \cdot A_e \tag{22}$$

The exact expressions, "Eqs. (20-22)", will be approximated by replacing the exact mean values of the temperature excess gradients at $A_{2,1}$ and A_e by

$$[\frac{d\bar{\theta}}{dx}]_{2,1} = \frac{\theta_2 - \theta_1}{L/N} \tag{23}$$

and

$$\left[\frac{d\bar{\theta}}{dx}\right]_{x=o} = \frac{\theta_1 - \theta_e}{L/2N} \tag{24}$$

The convective heat loss rate from CV_1 will be approximated by

$$Q_{c1} = h_1 \theta_1 S_1 \tag{25}$$

in which h_1 is the mean value of the heat transfer coefficient given by "Eq. (6)" and θ_1 is the effective local temperature excess.

The approximations given by "Eqs. (23-25)" will be used in "Eqs. (19-22)" to derive the general expressions of the temperature excess coefficients C_e and C_1.

By means of "Eq. (22)" with "Eq. (24)" we can solve for θ_e in terms of θ_1, thereby defining C_e:

$$\theta_e = \frac{\theta_1}{1 + \frac{Bie}{2N}} = C_e \theta_1 \tag{26}$$

where the Biot number, $Bie = h_e L/k$, is defined with respect to the end film coefficient h_e, the fin length L and the fin thermal conductivity k. It can be seen that when the fin end is insulated, ($h_e = 0$ and $Bie = 0$), $C_e = 1$; also when $Bie = \infty$, $C_e = 0$.

The relationship between θ_1 and θ_2 can be derived by means of "Eq. (19)" with the approximations given by "Eqs. (23) and (25)", and using the relationship given by "Eq. (26)". Therefore,

$$\theta_1 = \frac{\theta_2}{1 + \frac{h_1 L}{kN} \frac{S_1}{A_{2,1}} + \frac{Bie}{N} \frac{A_e \cdot C_e}{A_{2,1}}} = C_1 \theta_2 \tag{27}$$

It can be seen that the temperature excess coefficient of CV_1 depends upon several geometric characteristics, several heat transfer characteristics, the downstream temperature excess coefficient and the number of control volumes selected for subdivision. For an insulated end condition, the third term in the denominator of "Eq. (27)" vanishes. It is also apparent that the third term of the denominator of C_1 will not be present in non-truncated fins.

Internal Temperature Excess Coefficients C_j

The relationship between the local temperature excess θ_j and the upstream temperature excess θ_{j+1} can be derived by means of a heat balance over the boundaries of the arbitrary internal control volume CV_j as illustrated in Figure 7. The heat balance gives:

$$Q_{j+1,j} = Q_{cj} + Q_{j,j-1} \tag{28}$$

The conduction flow rates into and out of CV_j are approximated by

$$Q_{j+1,j} = k\left[\frac{\theta_{j+1} - \theta_j}{L/N}\right] \cdot A_{j+1,j} \tag{29}$$

and

$$Q_{j,j-1} = k\left[\frac{\theta_j - C_{j-1}\,\theta_j}{L/N}\right] \cdot A_{j,j-1} \tag{30}$$

where the downstream temperature excess θ_{j-1} has been replaced by $C_{j-1}\theta_j$.

The convective heat loss rate from CV_j is approximated by

$$Q_{cj} = h_j \theta_j S_j \tag{31}$$

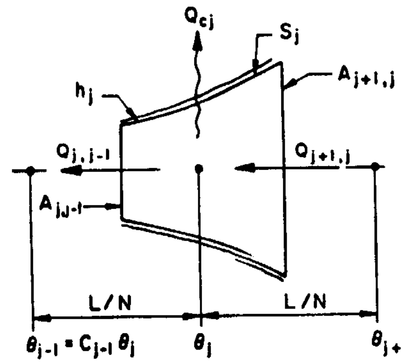

Fig. 7 Geometric and thermal relationships for arbitrary internal control volume

Solving for θ_j in terms of θ_{j+1} gives the temperature excess coefficient C_j:

$$\theta_j = \frac{\theta_{j+1}}{1 + \frac{h_j L}{kN} \frac{S_j}{A_{j+1,j}} + \frac{A_{j,j-1}}{A_{j+1,j}} (1 - C_{j-1})} = C_j \theta_{j+1} \tag{32}$$

This expression is valid for all internal control volumes for which $2 \leq j \leq (N-1)$.

Base Temperature Excess Coefficient C_N

The relationship between the local temperature excess θ_N and the fin base temperature excess θ_b' can be derived by means of a heat balance over the boundaries of the base control volume CV_N as illustrated in Figure 8. The heat balance gives

$$Q_b = Q_{cN} + Q_{N,N-1} \tag{33}$$

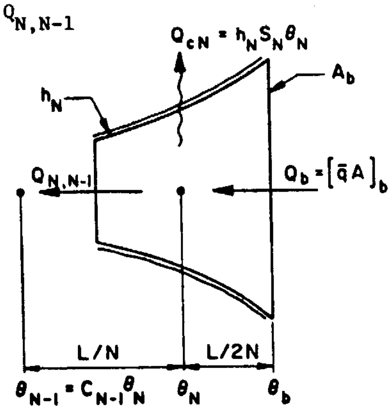

Fig. 8 Geometric and thermal relationships for base control volume

The conduction flow rates into and out of CV_N are approximated by

$$Q_b = k\left[\frac{\theta_b - \theta_N}{L/2N}\right] \cdot A_b \tag{34}$$

and

$$Q_{N,N-1} = k\left[\frac{\theta_N - C_{N-1}\theta_N}{L/N}\right] \cdot A_{N,N-1} \tag{35}$$

where the downstream temperature excess θ_{N-1} has been replaced by $C_{N-1}\theta_N$.

The convective heat loss rate from CV_N is approximated by

$$Q_{cN} = h_N \theta_N S_N \qquad (36)$$

Solving for θ_N in terms of θ_b gives the temperature excess coefficient C_N:

$$\theta_N = \frac{2\theta_b'}{2 + \frac{h_N L}{kN}\frac{S_N}{A_b} + \frac{A_{N,N-1}}{A_b}(1 - C_{N-1})} = C_N \theta_b' \qquad (37)$$

The base temperature excess coefficient C_N depends upon local geometric and heat transfer characteristics, the downstream temperature excess coefficient C_{N-1} and the number of control volumes.

Temperature Excess Coefficient for Imperfect Contact C_c

Whenever the extended surface (one-dimensional system) is mechanically attached or bonded to the base, there will be an additional temperature drop at the base due to the contact resistance. This interface resistance can be characterized by a contact resistance R_c or a contact conductance $h_c = (1/R_c A_b)$.

A heat balance across the base gives:

$$Q_b = h_c(\theta_b - \theta_b') A_b = \frac{\theta_b - \theta_b'}{R_c} = [k \frac{d\bar{\theta}}{dx}]_{x=L} \cdot A_b \qquad (38)$$

where θ_b is the base temperature excess, θ_b' is the fin base temperature excess, and $\theta_b - \theta_b'$ is the temperature drop across the interface due to the imperfect contact. For perfect contact this temperature difference vanishes.

As above we will approximate the mean value of the temperature excess gradient over the base area A_b by

$$[\frac{d\bar{\theta}}{dx}]_{x=L} = \frac{\theta_b' - \theta_N}{L/2N} = \frac{\theta_b' - C_N\theta_b'}{L/2N} \qquad (39)$$

Substitution of "Eq. (39" into "Eq. (38)" and solving for θ_b' in terms of θ_b yields

$$\theta_b' = \frac{\theta_b}{1 + \frac{2N}{Bic}(1 - C_N)} = C_c \theta_b \qquad (40)$$

where C_c is the temperature excess coefficient due to imperfect contact at the base and $Bic = h_c L/k > 0$ is the contact Biot number based upon the contact conductance, fin length and conductivity. It can be seen that C_c depends upon Bic, C_N and N. For perfect contact, $h_c = \infty$, Bic $= \infty$, therefore $C_c = 1$ and $\theta_b' = \theta_b$.

The general expressions of the temperature excess coefficients: C_e, C_1, C_j, C_N, and C_c are applicable to all extended surfaces (or one-dimensional conduction with convection systems). For convenience the relationships are listed in Table 1.

General Expressions of Conduction Areas and Convection Surfaces

The temperature excess coefficients listed in Table 1 depend upon local conduction areas and convection surfaces. For convenient reference these geometric characteristics will be reported here for the longitudinal fins, spines and pins, and circular annular fins.

The fin half-thickness for truncated fins is given by:

$$y(x) = a + (b-a)\left(\frac{x}{L}\right)^{\mu} \qquad (41)$$

where a and b are the fin half-thickness at the fin end, x = 0, and the fin base, x = L, respectively.

Table 1 General Expressions of Temperature Excess Coefficients

$$C_e = \frac{1}{1 + Bie/2N} \qquad (0 \le Bie < \infty)$$

$$C_1 = \frac{1}{1 + \frac{h_1 L}{kN}\frac{S_1}{A_{2,1}} + \frac{Bie}{N}\frac{A_e C_e}{A_{2,1}}}$$

$$C_j = \frac{1}{1 + \frac{h_j L}{kN}\frac{S_j}{A_{j+1,j}} + \frac{A_{j,j-1}}{A_{j+1,j}}(1 - C_{j-1})} \qquad (2 \le j \le N-1)$$

$$C_N = \frac{2}{2 + \frac{h_N L}{kN}\frac{S_N}{A_b} + \frac{A_{N,N-1}}{A_b}(1 - C_{N-1})}$$

$$C_c = \frac{1}{1 + \frac{2N}{Bic}(1 - C_N)} \qquad (0 < Bic \le \infty)$$

Longitudinal Fins

The variable conduction area is therefore:

$$A(x) = 2wy(x) = 2w[a + (b-a)(\frac{x}{L})^{\mu}] \qquad (42)$$

where w is the fin width.

The end and base conduction areas are $A_e = 2wa$ and and $A_b = 2wb$.

The conduction areas of the jth control volume CV_J are:

$$A_{j,j-1} = A_b\{\varepsilon + (1-\varepsilon)\ (j-1)^{\mu}/N^{\mu}\} \qquad (43)$$

and

$$A_{j+1,j} = A_b\{\varepsilon + (1-\varepsilon) j^{\mu}/N^{\mu}\} \qquad (44)$$

where $\varepsilon = a/b < 1$ is the fin half-thickness ratio.

In general the convection surfaces are given by

$$S = 2w \int_0^L \sqrt{1 + (dy/dx)^2}\, dx \qquad (45)$$

and

$$S_j = 2w \int_{(j-1)L/N}^{jL/N} \sqrt{1 + (dy/dx)^2}\, dx \qquad (46)$$

where

$$dy/dx = \mu(1-\varepsilon)(b/L)(x/L)^{\mu-1} \qquad (47)$$

For thin fins $b/L << 1$, $(dy/dx)^2 << 1$ and the convection surfaces reduce to $S = 2wL$ and $S_j = 2wL/N$, independent of the fin profile parameter μ.

Spines and Pins

The fin half-radius y(x) for a truncated fin is given by "Eq. (41)". The variable conduction area is therefore:

$$A(x) = \pi y^2 = A_b\{\varepsilon + (1-\varepsilon)(x/L)^{\mu}\}^2 \qquad (48)$$

where $A_b = \pi b^2$.

The conduction areas of the jth control volume

CV_j are

$$A_{j,j-1} = A_b \{\varepsilon + (1-\varepsilon)(j-1)^{\mu}/N^{\mu}\}^2 \quad (49)$$

and

$$A_{j+1,j} = A_b\{\varepsilon + (1-\varepsilon)j^{\mu}/N^{\mu}\}^2 \quad (50)$$

In general the convection surfaces are given by

$$S = \int_0^L 2\pi y \sqrt{1 + (dy/dx)^2}\, dx \quad (51)$$

and

$$S_j = \int_{(j-1)L/N}^{jL/N} 2\pi y \sqrt{1 + (dy/dx)^2}\, dx \quad (52)$$

For the special case of thin fins $(dy/dx)^2 << 1$, "Eqs. (51) and (52)" reduce to:

$$S = 2\pi bL \{\varepsilon + (1-\varepsilon)/(1+\mu)\} \quad (53)$$

and

$$S_j = 2\pi bL \left\{ \frac{\varepsilon}{N} + \frac{(1-\varepsilon)}{(1+\mu)N^{\mu+1}} \cdot [j^{\mu+1} - (j-1)^{\mu+1}] \right\} \quad (54)$$

Circular Annular Fins

The pertinent geometric parameters of circular annular fins are shown in Figure 9. The inner and outer fin radii relative to the tube axis are α and β respectively, and the fin length is $L = \beta - \alpha$. The fin half-thickness is given by "Eq. (41)" in which a and b are the fin half-thickness at the end and base of the fin. The cartesian coordinate x, whose origin is located in the fin end, is related to the radial coordinate r by

$$x = \beta - r \quad (55)$$

The variable conduction area is therefore

$$A(x) = 4\pi (\beta - x)\, y \quad (56)$$

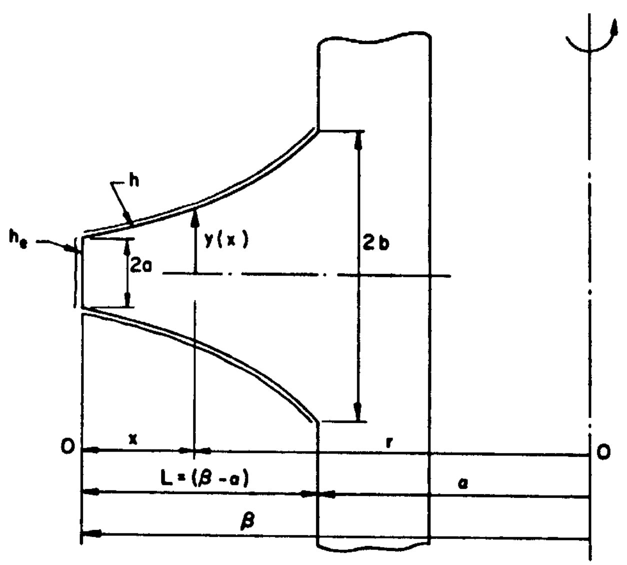

Fig. 9 Geometric parameters of truncated annular fin with coordinates

The end and base conduction areas are $A_e = 4\pi\beta a$ and $A_b = 4\pi\alpha b$.

In the general case the convection surfaces are given by

$$S = \int_0^L 4\pi(\beta - x) \sqrt{1 + (dy/dx)^2}\, dx \quad (57)$$

and

$$S_j = \int_{(j-1)L/N}^{jL/N} 4\pi(\beta - x) \sqrt{1 + dy/dx)^2}\, dx \quad (58)$$

where dy/dx is given by "Eq. (47)".

For the special case of thin fins, $(dy/dx)^2 << 1$, "Eqs. (57) and (58)" reduce to:

$$S = 2\pi(\beta^2 - \alpha^2) \quad (59)$$

independent of μ, and

$$S_j = \frac{2\pi\beta^2\rho}{N^2} \{2N - (2j - 1)\rho\} \quad (60)$$

with $\rho = 1 - \frac{\alpha}{\beta}$.

The circular annular fin numerical formulations can be used to approximate the numerical solution of longitudinal fins. This is accomplished by placing the circular fins on tubes whose outer radius α is some fraction of the fin outer radius β such that ρ is very nearly zero. The two constraints on α and β are therefore,

$$L = \beta - \alpha = \beta\rho \quad (61a)$$

$$\alpha/\beta \geq 0.99 \quad (61b)$$

The total heat flow rates through circular annular fins therefore approximate the total heat flow rates through longitudinal fins of width

$$w = \pi(\alpha+\beta) = \pi L\, (2 - \rho)/\rho \quad (62)$$

Nonuniform Heat Transfer Coefficient

In many practical fin problems, the lateral heat transfer coefficient is nonuniform (7, 8) over the fin length. The magnitude of the heat transfer coefficient is usually small (or zero) at the fin base; it increases with distance from the base, reaching a maximum value at the fin end (7).

If the heat transfer coefficient distribution over the lateral surface is known, its mean value over the fin length or over each control volume can be evaluated analytically or numerically.

To illustrate how this novel, numerical method can be used to handle nonuniform heat transfer coefficients, the power series form proposed by Han and Lefkowitz (7) will be applied to the longitudinal fin of arbitrary profile. The local value of the heat transfer coefficient is

$$h(x) = \bar{h}(1+\nu)\, [1 - \frac{x}{L}]^{\nu} \quad (63)$$

where ν is the nonuniform heat transfer coefficient parameter, and $\bar{h}$ is the overall mean value of h(x) defined by "Eq. (6)". When $\nu = o$, the coefficient is uniform over the lateral surface. When $\nu = 1$, the coefficient increases linearly from the fin base to its end. In all cases of $\nu > o$, the heat transfer coefficient is zero at the fin base.

The mean value of h(x) over the convection surface of the jth control volume is defined by

$$h_j = \frac{N}{L} \int_{(j-1)L/N}^{jL/N} \bar{h}(1+\nu)(1 - \frac{x}{L})^{\nu}\, dx \quad (64)$$

with $1 \le j \le N$. "Eq. 64)" can be integrated to give

$$h_j = \frac{\bar{h}}{N^\nu} \{[N - j+1]^{\nu+1} - [N - j]^{\nu+1}\} \tag{65}$$

The mean value over each control volume can be substituted into the temperature excess coefficient listed in Table 1. The second term in the denominator of C_1, C_j and C_N is dependent upon h_j. By examination of the general expression appearing in "Eq. (32)", we find that

$$\frac{h_j L}{kN} \cdot \frac{S_j}{A_{j+1,j}} = \frac{\gamma}{N^{2+\nu}} \{ \frac{[N - j + 1]^{\nu+1} - [N - j]^{\nu+1}}{\varepsilon + (1-\varepsilon)j^\mu/N^\mu} \} \tag{66}$$

where the fin parameter γ is defined by

$$\gamma = \frac{\bar{h}L^2}{kb} = Bi(L/b)^2 \tag{67}$$

and $Bi = \bar{h}b/k$ is the fin Biot number which is defined in terms of the fin half-thickness at the base. For the one-dimensional conduction model to be applicable, the Biot number must be small, i.e. $Bi < 0.1$.

The conduction-convection parameter given by "Eq. (66)" is valid for all longitudinal fins of arbitrary profile having arbitrary heat transfer coefficient. For this general case, the total heat loss rate from the fin becomes

$$Q = \frac{2wL\bar{h}}{N^{1+\nu}} \sum_{j=1}^{N} \theta_j \{[N-j+1]^{1+\nu} - [N-j]^{1+\nu}\} + h_e\theta_e A_e \tag{68}$$

Applications of the Temperature Excess Coefficient Method

To demonstrate the simplicity and accuracy of numerical computations by means of the temperature excess coefficient method, we will consider some examples appearing in (6).

The first examples consist of four non-truncated (ε=0), longitudinal, thin fins with perfect base contact ($h_c = \infty$, $Bie = \infty$, $C_c = 1$), uniform lateral heat transfer coefficient ($\nu = 0$), and insulated end ($h_e = 0$, $Bie = 0$, $C_e = 1$) when the fin is rectangular ($\mu = 0$). The thermal, heat transfer and geometric characteristics of the fins are listed in Table 2.

It can be shown that $Bi = 6.98 \times 10^{-3} << 0.1$ and $L/b = 21.33 >> 1$, therefore the thin fin model can be employed. The ideal heat loss rate from the fins is $Q_i = 2wLh\theta_b = 154.75$ W.

Table 2 Longitudinal Fin Characteristics (6)

T_b = 212°F (373 K)	k = 19.7 BTU/hrft°F (34.10W/mK)
T_f = 122°F (323 K)	h = 8.8 BTU/hrft^2°F (49.97W/m^{2}K)
θ_b = 90°F (50 K)	w = 1 ft (30.48 cm)
2b = 3/8 in (9.53 mm)	$\gamma = Bi(L/b)^2 = 3.175$
L = 4 in (10.16 cm)	$mL = \sqrt{\gamma}$

To compute the effective temperature excess of each control volume of these longitudinal fins we first require the general expression of the temperature excess coefficients. All coefficients C_j $(1 \le j \le N)$ can be computed directly by means of the following single, general expression:

$$C_j = \frac{\lambda}{\lambda + \frac{\gamma}{j^\mu N^{2-\mu}} + \xi(\frac{j-1}{j})^\mu (1 - C_{j-1})} \tag{69}$$

where $\xi = 0$ for $j = 1$ and $\xi = 1$ for $j \ge 2$; $\lambda = 1$ for $1 \le j \le N-1$ and $\lambda = 2$ for $j = N$. The temperature excess coefficients C_c and C_e are not required in these examples for the numerical computations of the total fin heat loss rates and fin efficiencies.

To compute the effective temperature excess at each control volume, put $\theta_j = C_j\theta_{j+1}$, with $1 \le j \le N$, beginning with $\theta_N = C_N\theta_b$. Having computed all temperature excess θ_j, we now proceed to compute the total fin heat loss rate:

$$Q = 2wLh \frac{1}{N} \cdot \sum_{j=1}^{N} \theta_j = 2wLh\bar{\theta} \tag{70}$$

where $\bar{\theta}$ is the mean effective fin temperature excess. We can normalize $\bar{\theta}$ by means of the fin base temperature excess; therefore "Eq. (70)" becomes

$$Q = 2wLh\theta_b \cdot \frac{\bar{\theta}}{\theta_b} = Q_i\eta \tag{71}$$

The fin efficiencies η can be computed by means of the following expression:

$$\eta = \frac{1}{N} \sum_{j=1}^{N} \frac{\theta_j}{\theta_b} \tag{72}$$

which represents the mean normalized control volume temperature excess. Employing the temperature excess coefficients, "Eq. (72)" can be re-written in nested-notation to show clearly the non-iterative aspect of the temperature excess coefficient method,

$$\eta N = (_N \ldots (((C_1+1)C_2+1)C_3+1)C_4+\ldots+1)C_{N-2}+1)C_{N-1}+1)C_N \tag{73}$$

The rapid convergence and the accuracy of the numerical values are clearly demonstrated in Table 3 for the four fin profiles studied by Kern and Kraus(6).

The difference between the analytical and numerical values decreases rapidly with increasing N. For these four examples, the five control volume subdivision gives reasonable accuracy (a relative error less than 2 percent) and the ten control volume subdivision gives excellent agreement with the analytical values.

As another example of the application of the temperature excess coefficient method, we will examine the case of longitudinal, thin fins of rectangular profile ($\mu = 0$), with insulated end ($h_e = 0$, $Bie = 0$, $C_e = 1$), perfect base contact ($h_c = \infty$, $Bic = \infty$, $C_c = 1$) and nonuniform heat transfer coefficient given by "Eq. (63)" [7]. Kern and Kraus [6] give the analytical solution for the case in which $\nu = 4$ and $\bar{h} = 49.97$ W/m^{2}K. The fin thermal conductivity, and its thickness and width are identical to the fins described in Table 2. For the fin length L = 7.62 cm, the fin Biot number is $6.98 \times 10^{-3} << 0.1$, and $L/b = 16 >> 1$; therefore the thin fin model is applicable. In this example the fin parameter $\gamma = Bi(L/b)^2 = 1.7868$. The ideal fin heat loss rate and fin efficiency are (6) $Q_i = 116.06$ W and $\eta = 0.429$.

A single, general expression of the temperature excess coefficients C_j $(1 \le j \le N)$ and the fin efficiency for the longitudinal, thin fin of arbitrary profile, arbitrary heat transfer coefficient, perfect base contact and non-truncated are listed in Table 4. For completeness and easy reference, expressions for the spines and pins, and circular annular fins are also listed in Table 4.

For the special case of $\mu = o$, $\nu = 4$, $Bie = o$, $Bic = \infty$, and $\gamma = 1.7868$ the temperature excess coefficients and fin efficiency can be computed by means of the following expressions:

Table 3 Convergence and Accuracy of the Numerical Results of the Temperature Excess Coefficient Method

Rectangular profile $\mu = 0$

N	Q[W]	η	% Difference on η and Q
3	78.46	.507	-4.34
5	80.62	.521	-1.70
10	81.71	.528	-0.38
Exact [6]	82.02	.530	-

Triangular profile $\mu = 1$

N	Q[W]	η	% Difference on η and Q
3	71.34	.461	-2.74
5	72.58	.469	-1.05
10	73.20	.473	-0.21
Exact [6]	73.35	.474	-

Convex parabolic profile $\mu = 1/2$

N	Q[W]	η	% Difference on η and Q
3	74.90	.484	-3.78
5	76.60	.495	-1.59
10	77.38	.500	-0.60
Exact [6]	77.84	.503	-

Concave parabolic profile $\mu = 2$

N	Q[W]	η	% Difference on η and Q
3	65.15	.421	-0.94
5	65.61	.424	-0.24
10	65.77	.425	0
Exact [6]	65.77	.425	-

Table 4 Temperature Excess Coefficients and Fin Efficiencies of Non-Truncated Longitudinal Fins, Spines and Pins, and Circular Annular Fins of Arbitrary Profile, Nonuniform Heat Transfer Coefficient, and Perfect Base Contact

Longitudinal Fins

$$C_j = \frac{\lambda}{\lambda + \frac{\gamma}{j^{\mu}\, N^{2+\nu-\mu}} \{[N-j+1]^{\nu+1} - [N-j]^{\nu+1}\} + \xi\left(\frac{j-1}{j}\right)^{\mu} (1-C_{j-1})}$$

$$\eta = \sum_{j=1}^{N} \frac{\theta_j}{\theta_b} \{[N-j+1]^{\nu+1} - [N-j]^{\nu+1}\}/N^{1+\nu}$$

Spines and Pins

$$C_j = \frac{\lambda}{\lambda + \frac{2\gamma}{(1+\mu)N^{2+\nu-\mu}} \left\{[N-j+1]^{\nu+1} - [N-j]^{\nu+1}\right\} \left\{\frac{j^{\mu+1} - (j-1)^{\mu+1}}{j^{2\mu}}\right\} + \xi\left(\frac{j-1}{j}\right)^{2\mu} (1-C_{j-1})}$$

$$\eta = \sum_{j=1}^{N} \frac{\theta_j}{\theta_b} \{[N-j+1]^{\nu+1} - [N-j]^{\nu+1}\} \{j^{\mu+1} - (j-1)^{\mu+1}\}/N^{1+\nu+\mu}$$

Circular Annular Fins

$$C_j = \frac{\lambda}{\lambda + \frac{\gamma}{2j^{\mu}\, N^{2+\nu-\mu}} \cdot \frac{2N-(2j-1)\rho}{N-j\rho} \{[N-j+1]^{\nu+1} - [N-j]^{\nu+1}\} + \xi\left(\frac{j-1}{j}\right)^{\mu} \frac{N-(j-1)\rho}{N-j\rho} (1-C_{j-1})}$$

$$\eta = \sum_{j=1}^{N} \frac{\theta_j}{\theta_b} \{[N-j+1]^{\nu+1} - [N-j]^{\nu+1}\} \{2N - (2j-1)\rho\}/N^{2+\nu}(2-\rho)$$

with $\rho = 1 - \frac{\alpha}{\beta}$

For all fins, $\xi = 0$ when $j = 1$ and $\xi = 1$ for $j \geq 2$; $\lambda = 1$ for $1 \leq j \leq N-1$ and $\lambda = 2$ for $j = N$

$$C_j = \frac{\lambda}{\lambda + \frac{1.7868}{N^6}\{[N-j+1]^5 - [N-j]^5\} + \xi(1 - C_{j-1})} \tag{74}$$

and

$$\eta = \frac{1}{N^5}\sum_{j=1}^{N}\frac{\theta_j}{\theta_b}\{[N-j+1]^5 - [N-j]^5\} \tag{75}$$

with $\theta_j = C_j\theta_{j+1}$, $1 \leq j \leq N$ and $\xi = 0$ at $j = 1$, $\xi = 1$ at $j \geq 2$; $\lambda = 1$ at $1 \leq j \leq N - 1$ and $\lambda = 2$ at $j = N$.

The computed fin efficiency with N = 5 is 0.431 which is 0.47% greater than the analytical value reported in (6). The agreement between the numerical value computed by means of the temperature excess coefficient method and the analytical value is excellent.

It can be shown that the computed values of the fin efficiencies are 0.434 and 0.430 for N = 3 and 10 respectively. These numerical values are within 1.17 and 0.23 percent of the analytical value. The accuracy of the temperature excess coefficient method is quite good even when the least number (N = 3) of control volumes is used.

To further demonstrate the rapid convergence of the temperature excess coefficient method, the constant cross-section fins with variable heat transfer coefficient studied by Han and Lefkowitz (7) were examined. The general longitudinal fin expressions listed in Table 4 were used. The fin profile parameter $\mu = o$; the heat transfer coefficient parameter ν was set to values of 0, 0.25, 0.50, 1.0 and 2.0. The fin parameter γ ranged between 0.25 and 16. The tolerance on the fin efficiency was set to 0.001. The computed values of the fin efficiencies ranged from 0.111 to 0.924. The number of control volumes required to achieve the prescribed tolerance ranged from a minimum of 6 to a maximum of 48. The minimum number is required when γ is small and ν is large; on the other hand the maximum number is required when γ is large and $\nu = o$.

The numerical values of η computed by means of the temperature excess coefficient method are in excellent agreement with the reported analytical values (7).

In all cases the difference between the numerical values of η computed by this novel method and those reported (7) is less than 1%.

Numerous other examples have been studied by many students over the past seven years. In all cases it was observed that the method proposed in this paper is easy to implement on microcomputers (11) and programmable calculators (12). The direct computations were found to be stable and accurate, and in most cases fewer than twenty control volumes are required to satisfy reasonable tolerances on temperature excess, total heat loss rate or fin efficiency.

Recently Zwart (13) was able to demonstrate by analysis that the temperature excess coefficient method (TECM) developed here is a special case of the so-called TDMA (Tri Diagonal - Matrix Algorithm) (14) which is a special case of the standard Gaussian-elimination method when the coefficient matrix is in the form of a tri-diagonal matrix.

SUMMARY

A novel, non-iterative, numerical method called the temperature excess coefficient method (TECM) has been developed for the solution of one-dimensional conduction with convection problems.

A general methodology has been presented for deriving, by means of heat balances over boundary and internal control volumes, general expressions of the temperature excess coefficients, temperature excess, total heat flow rate and fin efficiency for thin, truncated fins of arbitrary profile, nonuniform heat transfer coefficient, with end cooling and imperfect base contact. General expressions are given for longitudinal fins, spines and pins, and circular annular fins.

The simplicity of formulation, ease of programming, accuracy and rapid convergence of the numerical values are demonstrated by means of several fin examples given by Kern and Kraus (6). Few (N < 10) control volumes are required to satisfy reasonable convergence tolerances which may be put on i) the local temperature excess, ii) local temperature, iii) total heat flow rate or iv) the fin efficiency. In many cases as few as 3 to 6 control volumes are required to give reasonable approximation of fin efficiency. It was observed that the largest relative error occurs at the base temperature excess θ_N because of the temperature excess gradient approximation appearing in "Eq. (34)". This error can be minimized by making the base control volume smaller than the other control volumes; thereby placing θ_N closer to θ_b. But this expected increase in accuracy would be gained by some loss in the simplicity of the temperature excess coefficient formulation.

The general expressions of the temperature excess coefficients are shown to be dependent upon several dimensionless geometric and heat transfer parameters (L/b, μ, ν, Bi, Bie, Bic).

The temperature excess coefficient and fin efficiency expressions developed for circular annular fins can be used to obtain solutions for longitudinal fins; this is achieved by letting the fin radii ratio ρ approach zero.

The spine and longitudinal fin formulations yield the same numerical results for uniform cross-sections.

The temperature excess coefficient method can be extended to thick fins (dy/dx > 0.3); this would require analytical or numerical integration of the convection surface expressions.

ACKNOWLEDGEMENTS

The author thanks the graduate students of MIT (1976), of the Naval Postgraduate School (1980) and those of the University of Waterloo (since 1977) who have demonstrated by numerous examples the simplicity and accuracy of the method.

The author thanks J.P. Elliott for developing an interactive computer code in BASIC for the Apple II microcomputer based upon the method developed here and J. Zwart who showed that the method can be programmed on the HP-34C Programmable Calculator.

The interest and encouragement of Dr. C. E. Hermance in the exploration of the method and demonstrating that it can be applied to the bending of beams is greatly appreciated.

Finally, the author thanks the Canadian National Science and Engineering Council for financial support under operating grant A7445.

REFERENCES

1. Chapman, A.J., Heat Transfer, 3rd ed., The Macmillan Company, New York, 1974.
2. Karlekar, B.V., and Desmond, R.M., Heat Transfer, 2nd ed., West Publishing Company, St. Paul, 1982.
3. Lienhard, J.H., A Heat Transfer Text Book, Prentice-Hall, Inc., Englewood Cliffs, 1981.

4. Myers, G.E., Analytical Methods in Conduction Heat Transfer, McGraw-Hill Book Company, New York, 1971.
5. Rohsenow, W.M., and Choi, H., Heat, Mass, and Momentum Transfer, Prentice-Hall Inc., Englewood Cliffs, 1961.
6. Kern, D.Q., and Kraus, A.D., Extended Surface Heat Transfer, McGraw-Hill Book Company, New York, 1972.
7. Han, L.S., and Lefkowitz, S.G., "Constant Cross-Section Fin Efficiencies for Nonuniform Surface Heat-Transfer Coefficients," ASME Paper 60-WA-41, 1960.
8. Chen, S-Y, and Zyskowski, G.L., "Steady-State Heat Conduction in a Straight Fin with Variable Film Coefficient", ASME Paper 63-HT-12, 1963.
9. Smith, P.J., and Sucec, J., "Efficiency of Circular Fins of Triangular Profile", ASME Journal of Heat Transfer, Vol. 91, 1969, pp. 181-182.
10. Heggs, P.J., Ingham, D.B., and Manzoor, M., "The Effects of Nonuniform Heat Transfer from an Annular Fin of Triangular Profile", ASME Journal of Heat Transfer, Vol. 103, 1981, pp. 184-185.
11. Elliott, J.P., ME 482 Project, Department of Mechanical Engineering, University of Waterloo, 1981.
12. Zwart, J., "Generalized Non-Iterative Finite Difference Equations for an HP-34C Programmable Calculator", ME 651 Project, Department of Mechanical Engineering, University of Waterloo, 1982.
13. Zwart, J., Personal Communication.
14. Patankar, S.V., Numerical Heat Transfer and Fluid Flow, Hemisphere Publishing Corporation, New York, 1980.

HTD-Vol. 28

Heat Transfer in Electronic Equipment - 1983

presented at

THE WINTER ANNUAL MEETING OF
THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS
BOSTON, MASSACHUSETTS
NOVEMBER 13-18, 1983

sponsored by

THE K-12 COMMITTEE ON AIRCRAFT AND
ASTRONAUTICAL HEAT TRANSFER
THE HEAT TRANSFER DIVISION, ASME

edited by

S. OKTAY
IBM HOPEWELL JUNCTION

A. BAR-COHEN
BEN GURION UNIVERSITY

THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS
United Engineering Center 345 East 47th Street New York, N.Y. 10017