CONTACT CONDUCTANCE CORRELATIONS
OF ELASTICALLY DEFORMING FLAT ROUGH SURFACES

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Abstract

In many engineering applications it is of paramount importance to be able to predict the thermal conductance of surfaces in contact. These predictions must be accurate and preferably contain a minimum number of surface roughness parameters. The work presented here is an extension of Greenwood's (1966) analysis and is aimed at providing engineering correlations for the prediction of contact conductance in rough but flat surfaces where the deformation is elastic. All rough flat surfaces must at some point in time be bound by this mode of deformation even though initially the contacting asperities deform in a nonelastic manner.

In the analysis presented here, the surface roughness was not restricted to any particular distribution and the asperities were assumed to have constant peak radii. In the thermal modelling the effect of constriction is included.

To make the analysis useful for practicing engineers, correlations were obtained for nearly Gaussian distributions.

A comparison between correlation resulting from this work and those of previous investigations indicated the following;

- Mikic (1974), on the basis of previous work by Cooper, Mikic and Yovanovich (1969) and by assuming that the elastic contact area is exactly half the plastic contact area, derived the following expression;

$$h_c^* = 3.6 \left(\frac{\bar{p}}{E'}\right)^{.94} (\tan\theta)^{.06}$$

- Bush and Gibson (1979), assuming a Gaussian distribution of the surface roughness and a variable peak radius, suggest that a good correlation of their results is;

$$h_c^* = 1.69 \left(\frac{\bar{p}}{E'}\right)^{.89} \sqrt{\sigma} \; m_4^{\frac{1}{4}} (\tan\theta)^{-.89}$$

- In the work presented here it was found that

$$h_c^* = 3.81 \left(\frac{\bar{p}}{E'}\right)^{.93} \left(\frac{\sigma}{\beta}\right)^{.035}$$

where hc* = nondimensionalized thermal conductance = $h_c\sigma/k_m$
$\bar{p}$ = mean pressure
E' = composite elastic modulus
σ = composite RMS roughness
β = mean asperity peak radius
$\tan\theta$ = mean slope of the asperities
m_4 = the fourth moment of the surface roughness

All the investigations included the effect of constriction.

An examination of the above correlations reveals that the one due to Mikic (1974) and the one presented in this paper are almost identical and nearly insensitive to secondary surface roughness parameters. This is not so clear from the expression by Bush and Gibson (1979). Insensitivity to secondary surface roughness parameters is very desirable because the majority of industry does not have ready access to these parameters.

The influence of the thermal constriction may be observed from the work of Greenwood (1966) and the current paper. Greenwood (1966) did not consider thermal constriction in his analysis and this shows up through a lower dependence of load on the contact conductance.

Nomenclature

A	Hertzian contact area
Ā	Apparent contact area
A_r	Real contact area
a	Circular microcontact radius
b	Circular heat flux tube radius
E	Elastic modulus
E'	Equivalent elastic modulus

$$E' = \left(\frac{1-\nu 1}{E_1} + \frac{1-\nu 2^2}{E_2}\right)^{-1}$$

F	Contact load
hc	Unit surface conductance
hč	Nondimensional unit surface conductance
k	Thermal conductivity
k_m	Harmonic mean thermal conductivity
N	Asperity density on a rough surface
n	Number of contacting asperities
$\bar{p}$	Mean contact pressure
Rc	Thermal constriction resistance
Sc	Thermal constriction parameter
z	A coordinate perpendicular to the contact plane

Greek Symbols

β	Asperity tip radius
σ	Standard deviation of the asperity height distribution
η	Contacting asperity density within the apparent area
δ	Compliance
ν	Poisson's ratio
ε	Ratio of microcontact radius to corresponding heat flux tube radius
μ_n	Integral of the Gaussian distribution of order n.

Subscripts

1,2	Referring to two solids in a contact
i	Referring to an individual asperity

Physical Model

Real surfaces, when viewed under high magnification consist of a multitude of apparently random peaks and valleys (Greenwood, 1964). Initial contact between two such surfaces occurs just at the highest asperities while continued loading causes existing microcontacts to grow in size and new sites to develop.. The mode of deformation of any asperity may be either elastic or plastic depending upon several parameters. At all times asperity deflection is assumed to occur independently and no microcontact coalescence is possible. All loading conditions produce nearly constant average contact size allowing a quasi-direct relation to occur between the real contact area and the load (Archard,1953) a necessary consideration to satisfy the classic laws of friction (Bowden and Tabor, 1954).

To mathematically model contact mechanics with two rough surfaces is unnecessarily complex, and following with Greenwood and Tripp (1971) it is possible to model any real contact as one between a smooth, nominally flat, elastic halfspace and an elastic surface with some equivalent geometry and roughness. The asperity peaks are assumed to be distributed according to a Gaussian distribution about some mean reference plane and are hemispherically dome shaped near their tips. Although it seems intuitive to include some dependence of asperity tip radii with asperity height (Mikic, 1973), the authors believe it reasonable to assume that the asperity peak radii, β, are constant (Greenwood,1964).

In the past, authors (Mikic, 1973; Cooper, Mikic and Yovanovich, 1968) have described real surfaces by the mean slopes of their roughness profiles and through mathematical expressions such as moments of the power spectral density of a surface profile (Bush and Gibson, 1979). It should be remembered that the aim of this development is to derive an expression allowing a unit surface conductance to be easily calculable. Thus the parameters used in this exercise must be readily obtainable. Therefore it is decided that a real surface may be described by the standard deviation of asperity peaks, σ, the constant tip radius, β, and N, the surface density of asperities. The authors realize that the complete surface is being described only by the profile information near the peaks and this may be justified as this is the portion of the real surface which is in actual contact.

A characteristic reference plane exists through the rough surface from which the surface height, z, may be continuously measured. Additionally, d, defines the proximity of the reference plane to the smooth flat surface (see Fig. 1).

All microcontacts are circular, as they are formed from hemispherical asperity peaks and they may be of different radii depending upon the extent of asperity deformation. Hertizian theory is applied to the contact of individual asperities and Timoshenko and Goodier (1951) present the associated theories of elasticity to determine relations for the area of contact, A, the radius of the contact area, a, and the load supported, Fi, in terms of δ, the asperity compliance (defined as the distance by which points far from the contact zone approach each other during the contact).

$$A = \pi\beta\delta \tag{1}$$

$$a = \sqrt{\beta\delta} \tag{2}$$

$$F_i = 2/3\ E'\ \beta^{1/2}\ \delta^{3/2} \tag{3}$$

If the asperity height distribution is represented by some statistical function ϕ_o, where $\phi_o(z)$ represents the probability of an asperity peak occuring between height z and z+dz, then, any asperity will be in contact if its undeformed height is greater than the distance between the reference plane and smooth halfspace. Hence the probability of a contact is given by

$$\text{Probability of a Contact} = \int_d^\infty \phi_o(z)\,dz \tag{4}$$

The number of contacting asperities existing for a specific approach of both bodies, with apparent contact area, A, given by

$$n = NA \int_d^\infty \phi_o(z)\,dz \tag{5}$$

Greenwood and Williamson (1966) have extended this concept and have performed the following basic substitutions.

Determine the asperity compliance to be:

$$\delta = z - d \tag{6}$$

The real area of contact, Ar, visualized as the sum of the individual microcontact areas is given by:

$$Ar = \pi\beta NA \int_d^\infty (z-d)\ \phi_o(z)\,dz \tag{7}$$

Nondimensionalizing the distance parameters with respect to σ,

$$\lambda = d/\sigma \tag{8}$$

$$h = z/\sigma \tag{9}$$

Now introduce a new term representing the integral of the asperity peak distribution of order n:

$$\mu_n(\lambda) = \int_\lambda^\infty (h-\lambda)^n\ \phi^*(h)\,dh \tag{10}$$

Thus the real area of contact may be written as:

$$Ar = \pi\beta NA\sigma\ \mu_1(\lambda) \tag{11}$$

and the load carried between two surfaces is:

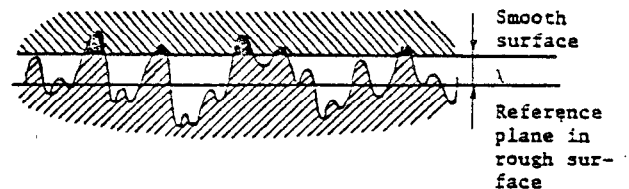$$F = 2/3\ E'NA\beta^{1/2}\ \sigma^{3/2}\ \mu_{3/2}(\lambda) \tag{12}$$

Figure 1. A Smooth Flat Halfspace in Contact with a Flat Rough Surface.

Thermal Model

Heat transfer is examined for engineering contact conditions in a vaccum where radiation is negligible. Thus, only direct transfer of heat by conduction through the discrete microcontacts is possible and to understand this further it is necessary to present the model of a heat flux tube.

Associated with each microcontact is a pair of abutting cylinders called heat flux tubes. Each is geometrically defined by cylindrically shaped closed boundaries extending from the contact into the body. These boundaries are adiabatic so that all heat transfer within each body flows through a bundle of tubes aligned with the flux. Each tube has an individual, equivalent radius, b_i, such that $b_i > a_i$ and thus the cross sectional area available to the flow of heat is choked at the microcontact creating the thermal constriction resistance. The authors admit that a paradox exists by using tubes of circular section rather than, say, hexagonal, as there will always be interstitial area over which no heat flux exists. This implies that the sum of the tube areas never does (but should) equal the apparent contact area.

Clearly, to develop equations describing the heat transfer phenomenon occuring here, it is first necessary to obtain the solution of the situation for one microcontact and tube pair. To do this, one must determine the axisymmetric steady state solution of the Laplace equation in cylindrical coordinates. This mixed boundary value problem is not easily solved in closed analytic form (Gibson, 1976) but infinite series-type solutions are common (Yovanovich, 1976 etc.). When one assumes that the microcontact interface is planar, that all microcontacts are isothermal and that heat transfer occurs for the case of concentric microcontact and heat flux tube, Yovanovich (1976) has shown the individual asperity constriction resistance within one body to be:

$$Rc_i = \frac{Sc_{1i}}{4k_1 a_i} \tag{13}$$

where k_1 = Thermal conductivity for solid 1

Sc_{1i} = a thermal constriction parameter for the ith contact in solid 1

$$= \frac{8}{\pi \epsilon_i} \sum_{n=1}^{\infty} \frac{J_1(\lambda_n b_i \epsilon_i) \sin(\lambda_n b_i \epsilon_i)}{(\lambda_n b_i)^3 J_0^2(\lambda_n b_i)}$$

However, approximate expressions for the thermal constriction parameter exist for certain ranges of a/b.

$$Sc_i = 1 - 1.4197 \frac{a_i}{b_i} \qquad 0 \le \frac{a}{b} \le 0.1 \tag{15}$$

$$Sc_i = (1 - \frac{a_i}{b_i})^{1.5} \qquad \frac{a}{b} \le 0.3 \tag{16}$$

The total resistance to heat flow from one heat tube, through the contact, to the adjacent tube may be determined by the sum of the two individual resistances.

$$Ro_i = Rc_{1i} + Rc_{2i} \tag{17}$$

It is reasonable to assume that the constriction parameters are equal on both sides of the contact resulting with:

$$Ro_i = \frac{Sc_i}{2k_m a_i} \tag{18}$$

where k_m = the harmonic mean conductivity

Amalgamation of the Physical and Thermal Models

Within the apparent contact area, if n asperities are touching, then the total thermal resistance is:

$$\frac{1}{Rc} = \sum_{i=1}^{n} \frac{1}{Ro_i} \tag{19}$$

For the particular case where all touching asperities have the same thermal constriction parameter and contact radius,

$$Rc = \frac{Ro_i}{n} = \frac{Sc}{2nk_m a} \tag{20}$$

which is an established relation by several authors

(See appendix A for further developments based on this assumption).

Alternatively, the situation where microcontacts have different radii should be closely examined. Using the number of contacting asperities as given by equation 5, it is possible to conclude that the unit surface conductance is given by:

$$hc = \frac{1}{A} \sum_{i=1}^{n} \frac{1}{Ro_i} = 2k_m N \int_{\lambda}^{\infty} \frac{\phi_0(z) a_i dz}{Sc_i} \tag{21}$$

where the summation of the discrete number of contact sites has been replaced by a continuous integral.

From simple geometric relations, it may be shown that the radius of the microcontact is expressed in terms of asperity tip radius and the compliance

$$a = \sqrt{\beta \delta} \tag{22}$$

or $\qquad a = \sqrt{\beta \sigma (\lambda - h)} \tag{23}$

Additionally, the mean pressure, $\bar{P}$, is defined as the total load, F, divided by the apparent area A. Substituting this with equations 5 and 21, the unit surface conductance is:

$$hc = \frac{3k_m \bar{P}}{\sigma Sc E'} \frac{\mu_{1/2}(\lambda)}{\mu_{3/2}(\lambda)} \tag{24}$$

Since the sum of the microcontact areas represents the real contact area and the sum of the heat flux tube cross-sectional areas represents the apparent contact area,

$$\frac{a}{b} = \frac{Ar}{A} \tag{25}$$

Through equation 11, the ratio of areas may be re-expressed so that

$$\frac{a}{b} = \sqrt{\pi \beta N \sigma \ \mu_1(\lambda)} \tag{26}$$

Substituting this and the simplified forms of the constriction parameter into equation 24 result with,

$$hc = \frac{3k_m \bar{P} \ \mu_{1/2}(\lambda)}{\sigma E' \mu_{3/2}(\lambda)\{1 - 1.4197[\pi \beta N \sigma \mu_1(\lambda)^{0.5}\}}, \ 0 \le \frac{a}{b} \le 0.1 \tag{27}$$

$$hc = \frac{3k_m \bar{P} \mu_{1/2}(\lambda)}{\sigma E' \mu_{3/2}(\lambda)\{1 - [\pi \beta N \sigma \mu_1(\lambda)]^{0.5}\}^{1.5}}, \ \frac{a}{b} \le 0.3 \tag{28}$$

It should be noted that the above expressions are not restricted to the assumption of any specific distribution of asperity peaks. Any representative mathematical function may be substituted to simplify the equations into tractable form.

Consider the Gaussian distribution of asperity peaks. It is convenient to define ζ such that

$$hc = \frac{3km\bar{P}}{\sigma E'} \zeta \qquad (29)$$

where $\zeta = \dfrac{\mu_{1/2}(\lambda)}{\mu_{3/2}(\lambda)\{1-1.4197[\pi c \mu_1(\lambda)]^{0.5}\}}$,

$$0 \leq \frac{a}{b} \leq 0.1$$

$$\zeta = \frac{\mu_{1/2}(\lambda)}{\mu_{3/2}(\lambda)\{1-[\pi c \mu_1(\lambda)]^{0.5}\}1.5},$$

$$\frac{a}{b} \leq 0.3$$

where Greenwood and Tripp (1971) have determined $N\delta\sigma = c$ a constant for a wide variety of experimental surfaces.

Using equation 12 it is possible to correlate the following by power law

$$\zeta = \alpha \left(\frac{3}{2c} \left(\frac{\bar{P}}{E'}\right) \frac{\delta}{\sigma}\right)^\gamma$$

for the practical range of $1.0 \leq \lambda \leq 2.5$

Table 1 Power law regression coefficients

	c=0.05		c=0.07		c=0.10	
	$0\leq\frac{a}{b}\leq0.1$	$\frac{a}{b}\leq0.3$	$0\leq\frac{a}{b}\leq0.1$	$\frac{a}{b}\leq0.3$	$0\leq\frac{a}{b}\leq0.1$	$\frac{a}{b}\leq0.3$
α	1.357	1.364	1.430	1.437	1.531	1.538
γ	−0.075	−0.078	−0.067	−0.070	−0.057	−0.059
r	0.9999	0.9997	0.9998	0.9999	0.9967	0.9977

This enables the unit surface conductance to be expressed in nondimensional form as summarized below.

Table 2 Simplified expressions for non-dimensional unit surface conductance

c	$0\leq\frac{a}{b}\leq0.1$	$\frac{a}{b}\leq0.3$
0.05	$3.15(\frac{\sigma}{\delta})^{0.038}(\frac{\bar{P}}{E'})^{0.925}$	$3.14(\frac{\sigma}{\delta})^{0.039}(\frac{\bar{P}}{E'})^{0.922}$
0.07	$3.49(\frac{\sigma}{\delta})^{0.034}(\frac{\bar{P}}{E'})^{0.933}$	$3.48(\frac{\sigma}{\delta})^{0.035}(\frac{\bar{P}}{E'})^{0.930}$
0.10	$3.94(\frac{\sigma}{\delta})^{0.029}(\frac{\bar{P}}{E'})^{0.943}$	$3.93(\frac{\sigma}{\delta})^{0.030}(\frac{\bar{P}}{E'})^{0.941}$

Conclusions

Elastic deformation of touching, flat, rough surfaces is considered where only heat conduction through the asperity tips is possible. When it is assumed that the asperity peaks are hemispherically shaped and are distributed in a Gaussian manner, then

the non-dimensionalized unit surface conductance is found to be

$$hc^* = 3.5(\frac{\sigma}{\delta})^{0.035}(\frac{\bar{P}}{E'})^{0.93}, \quad c=0.07$$

This shows the conductance to be almost proportional to the $(\bar{P}, E')$ ratio and nearly independent of the (σ/δ) quantity. Thus, in the dimensional form, the unit surface conductance is practically dependent upon only one parameter which describes the surface; σ, the standard deviation. This ability to eliminate other surface parameters is possible through establishing that $N\delta\sigma = c$, a constant of which the exact value is truly an arguable point. The authors, as yet, do not have any experimental confirmation of this and so must rely upon the works of Greenwood, Williamson, Tripp, Archard and Whitehouse. It should however be noted that for several values of c it is only the coefficient in the previous equation which varies while the exponents remain almost unchanged.

Additionally, if a further restriction of constant microcontact radius is imposed, then,

$$hc^* = 4.1(\frac{\sigma}{\delta})^{0.03}(\frac{\bar{P}}{E'})^{0.94}, \quad c = 0.07$$

showing that the fashion in which the microcontact radius varies over the apparent contact area does not strongly influence the conductance.

Mikic(1974) has examined a similar phenomenon and determined that

$$hc = 2.15 \frac{km}{\sigma} (\tan \theta)^{0.06} (\frac{\bar{P}}{E'})^{0.94}$$

However some of his further data allows the above expression to be written as:

$$hc^* = 2.1(\frac{\sigma}{\delta})^{0.03}(\frac{\bar{P}}{E'})^{0.94}$$

which shows striking similarities with the equations presented here.

Appendix A

For the particular case of constant microcontact radius for all contacts, the following analysis is conducted.

Combining equations 5 and 11 together,

$$\sqrt{\frac{A_r}{n\pi}} = \sqrt{\frac{\mu_1(\lambda)\delta\sigma}{\mu_0(\lambda)}}$$

Now, the real area of contact may be evaluated by the product of the number of contacts and the individual contact area,

$$A_T = n\pi a^2 \quad \text{or} \quad \sqrt{\frac{A_r}{n\pi}} = a$$

Thus equating the above two expressions,

$$a = \sqrt{\frac{\mu_1(\lambda)\delta\sigma}{\mu_0(\lambda)}}$$

Combining equations 5 and 12 and dividing by the apparent contact area A, a density of contacting asperities is found to be

$$\eta = \frac{n}{A} = \frac{3}{2} \frac{\bar{P}}{E'\delta^{1/2}\sigma^{3/2}} \frac{\mu_0(\lambda)}{\mu_{3/2}(\lambda)}$$

Copper, Mikic and Yovanovich (1968) have obtained an expression for unit surface conductance

$$h_c = \frac{2nk_m a}{S_c}$$

or

$$h_c = \frac{3\sqrt{u_0(\lambda)u_1(\lambda)}}{\mu_{3/2}(\lambda)} \frac{\bar{P}k}{S_c E'\sigma}$$

However, rewriting equation 12 and substituting in the above,

$$h_c = \frac{2\sqrt{u_0(\lambda)u_1(\lambda)}}{S_c} (N\beta\sigma) \frac{km}{3\sigma}$$

Now it is possible to express a new variable, Γ, where

$$\Gamma = \frac{\sqrt{u_0(\lambda)u_1(\lambda)}}{S_c} (N\beta\sigma)$$

Greenwood and Tripp (1971) have determined that $N\beta\sigma$ is approximately constant for a wide variety of engineering surfaces. Let this quantity equal c, a constant

$$N\beta\sigma = c$$

Including equations 15, 16 it is possible to write

$$\Gamma = c\sqrt{u_0(\lambda)\mu_1(\lambda)} \{1-1.4197(\pi c\mu_1(\lambda)^{0.5}\}^{-1.0}$$

$$\Gamma = c\sqrt{u_0(\lambda)\mu_1(\lambda)} \; 1-[\pi c\mu_1(\lambda)]^{0.5}\}^{-1.5},$$
$$\frac{a}{b} < 0.3$$

Surely Γ must be load dependent and to establish this relation, equation 12 is rearranged to give

$$\frac{\bar{P}}{E'}\sqrt{\frac{\beta}{\sigma}} = \frac{2}{3} c\mu_{3/2}(\lambda)$$

It is now possible to conduct a power law correlation over the range $1.0 \leq \lambda \leq 2.5$ of the form

$$\Gamma = \alpha\left(\frac{\bar{P}}{E'}\sqrt{\frac{\beta}{\sigma}}\right)^\gamma$$

where α, γ are given in table 3 for various values of c and based on both expressions for

Table 3 Power law regression coefficients

	c=0.05		c=0.07		c=0.10	
	$0\leq\frac{a}{b}\leq0.1$	$\frac{a}{b}\leq0.3$	$0\leq\frac{a}{b}\leq0.1$	$\frac{a}{b}\leq0.3$	$0\leq\frac{a}{b}\leq0.1$	$\frac{a}{b}\leq0.3$
α	1.815	1.816	2.064	2.049	2.424	2.368
γ	0.931	0.931	0.940	0.939	0.953	0.951
r	0.999	0.999	0.999	0.999	0.999	0.999

Consider the unit surface conductance to be expressed in dimensionless form

$$h_c* = \frac{h_c\sigma}{k_m} = 2\alpha\left(\frac{\beta}{\sigma}\right)^{(\gamma-1)/2}\left(\frac{\bar{P}}{E'}\right)^\gamma$$

Hence by substitution of the above numerical data it is possible to arrive at the following equations for non-dimensional unit surface conductance.

Table 4. Simplified expressions for nondimensional unit surface conductance.

c	$0\leq\frac{a}{b}\leq0.1$	$\frac{a}{b}\leq0.3$
0.05	$3.63(\frac{\sigma}{\beta})^{0.03}(\frac{\bar{P}}{E'})^{0.93}$	$3.63(\frac{\sigma}{\beta})^{0.03}(\frac{\bar{P}}{E'})^{0.93}$
0.07	$4.13(\frac{\sigma}{\beta})^{0.03}(\frac{\bar{P}}{E'})^{0.94}$	$4.10(\frac{\sigma}{\beta})^{0.03}(\frac{\bar{P}}{E'})^{0.94}$
0.10	$4.85(\frac{\sigma}{\beta})^{0.02}(\frac{\bar{P}}{E'})^{0.95}$	$4.74(\frac{\sigma}{\beta})^{0.02}(\frac{\bar{P}}{E'})^{0.95}$

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