THERMAL CONSTRICTION RESISTANCE DUE TO A CIRCULAR ANNULAR CONTACT

M.M. Yovanovich* and G.E. Schneider+

University of Waterloo, Waterloo, Ontario

Abstract

A formal, closed-form solution for the constriction resistance of a circular annular contact on a half-space has been obtained for the constant-flux boundary condition. The solution is valid for all values of the geometric parameter. A simple, approximate solution is presented for radii ratios very nearly unity. The results of the analysis are compared with the approximate solution of Smythe and the analytical-numerical solution of Cooke.

Nomenclature

$A(\lambda)$	=	function defined by Eq. (5)
a	=	inner radius of annular contact
b	=	outer radius of annular contact
C	=	capacitance, Eq. (1)
E	=	complete elliptic integral of the second kind
K	=	complete elliptic integral of the first kind
k	=	thermal conductivity
Q	=	total heat flow rate
q	=	heat flux
R	=	thermal constriction resistance, Eq. (12)
R^*	=	dimensionless constriction resistance (kbR)
T	=	temperature
$\overline{T}$	=	average contact area temperature, Eq. (7)
r,z	=	polar coordinates

Presented as Paper 76-142 at the AIAA 14th Aerospace Sciences Meeting, Washington, D.C., Jan. 26-28, 1976.

* Professor, Thermal Engineering Group, Department of Mechanical Engineering.

+ Project Engineer, Thermal Engineering Group, Department of Mechanical Engineering.

Γ = contact area
γ = dimensionless parameter (ε/κ)
ε = dimensionless radii ratio (a/b)
ε_o = dielectric constant
κ = dimensionless parameter (a/r), modulus of E and K
κ' = complementary modulus with respect to κ ($= \sqrt{1 - \kappa^2}$)
λ = variable in temperature distribution, Eq. (5)
ψ = constriction parameter (= dimensionless resistance)
χ = defined by Eq. (2)

Subscript

a = annular

Superscript

T = isothermal

Introduction

The thermal resistance across a joint consisting of a solid or hollow, metallic O-ring in elastic contact with two large smooth flats[1] is of some interest to aerospace thermal designers. The maximum resistance will occur in a vacuum environment when radiative transfer is negligibly small. The contact formed at the interface between the O-ring and one of the flats due to elastic deformations will be a circular annular area having radii a, b (a < b). It can be shown using elasticity theory that the ratio a/b will be very nearly unity in cases of practical interest. It is required to determine the thermal constriction resistance due to such a contact when there is no radiative transfer inside or outside the contact area.

It is well-known fact that one can use the analogy between electrostatics (or electrodynamics) and conduction to obtain the geometric parameter that determines the effect of the contact shape. For example, the capacitance of a solid circular disk of radius b in an infinitely extended medium is defined as[2]

$$\text{capitance (C)} = \frac{\text{total charge (Q) on the disk}}{\text{potential difference (V)}} \qquad (1)$$

The definition can be used to obtain the capacitance, which is found to be $C = 8b\varepsilon_o$[2], where ε_o is the absolute dielectric constant of free space (vacuum). It is also well known that the thermal constriction resistance of an isothermal, circular disk of radius b within an infinite medium of thermal conductivity k is[7] $R = 1/(8kb)$.

We see that both physical problems are mathematically equivalent and that the capacitance divided by the dielectric constant is equal to the reciprocal of the product of the constriction resistance and the thermal conductivity in the following manner:

$$C/\varepsilon_o = 1/kR = \chi a \qquad (2)$$

In Eq. (2), the parameter χ is a function of the geometry (shape of the disk) and the boundary condition. For the example just given, $\chi = 8$. By means of Eq. (2), one can always determine the thermal constriction resistance of a particular contact area given the capacitance of the same geometry having mathematically equivalent boundary conditions.

A survey of the open literature reveals that two pertinent papers[2,3] have considered the capacitance of a circular, annular geometry. In both studies, it was assumed that the annular disk had a uniform potential. Smythe[2] used the concept of superposition of known solutions to obtain an approximate solution to the capacitance problem. Cooke[3], on the other hand, obtained a formal solution to a set of triple integral equations that result from the use of circular cylinder coordinates to formulate the capacitance problem. The solution was obtained in terms of a function that was determined using a Fredholm integral equation of the first kind.

The results of these studies[2,3] can be used to determine the constriction resistance of an isothermal, circular, annular contact using Eq. (2). The purpose of this paper is to obtain a closed-form solution to the problem described herein for the case of a uniform heat flux boundary condition by means of the integral formulation[4] developed recently, and to compare these results with those of Smythe[2] and Cooke.[3]

Classical Problem Solution

Problem Statement

A circular, annular contact area of radii a, b $(a < b)$ is situated on the surface of a half-space of thermal conductivity k (Fig. 1). Heat enters the half-space through the contact area only and flows steadily through the conducting

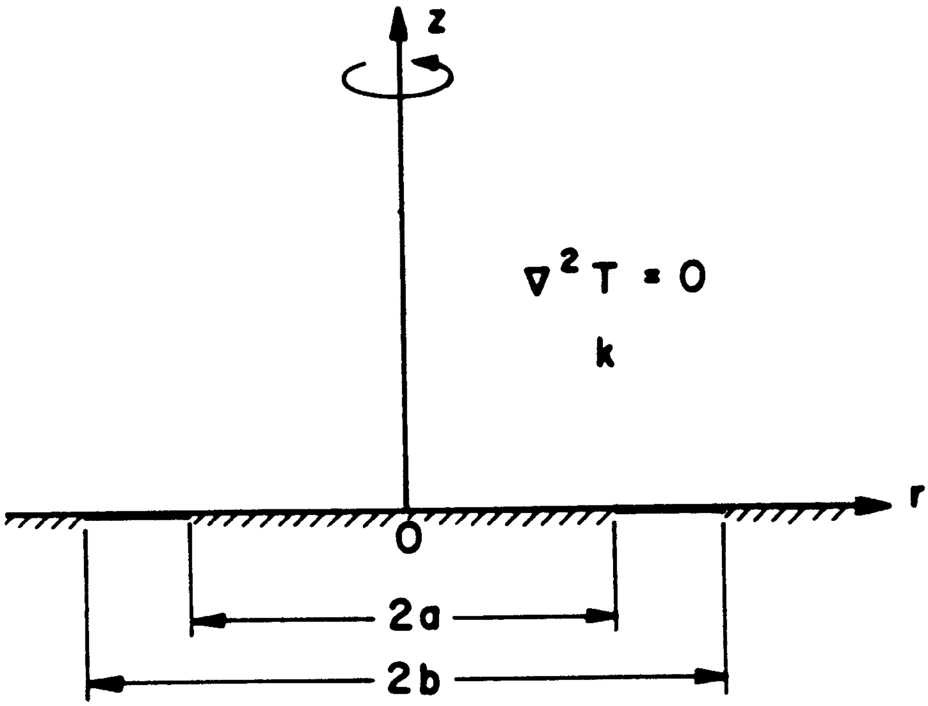

Fig. 1 Circular annular contact.

medium, leaving it through a boundary far removed from the contact area. The surface of the half-space within the inner radius a and outside the outer radius b is assumed to be impervious to heat transfer. Over the contact area, either the temperature or the flux may be prescribed (e.g., uniform). The temperature within the half-space tends toward a uniform temperature at points far from the contact area. For convenience, this temperature will be taken to be zero.

The thermal constriction resistance is defined as the average temperature of the contact area minus the temperature at infinity divided by the total heat flow rate through the contact area. In order to determine this resistance, we must obtain the temperature field and the heat flow rate.

Solution

The classical approach to this problem is to place the origin in the center of the annular contact (Fig. 1) and to write the governing differential equation in circular cylinder coordinates (r,z):

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{3}$$

It is clear that we are concerned with axisymmetric problems only.

The temperature field must satisfy the following boundary conditions:

$$z = 0,\ 0 \le r < a, \qquad \frac{\partial T}{\partial z} = 0$$

$$b < r, \qquad \frac{\partial T}{\partial z} = 0$$

$$a < r < b, \qquad T = T_o \text{ or}$$

$$-k\,\frac{\partial T}{\partial z} = \frac{Q}{\pi(b^2 - a^2)} \tag{4a}$$

$$r = 0,\ z \ge 0, \qquad \frac{\partial T}{\partial r} = 0 \tag{4b}$$

$$r \to \infty,\ z = 0, \qquad T \to 0 \tag{4c}$$

$$z \to \infty,\ r = 0, \qquad T \to 0 \tag{4d}$$

The solution of Eq. (3) satisfying boundary conditions (4b - 4d) is[2]

$$T(r,z) = \int_0^\infty A(\lambda)\, e^{-\lambda z}\, J_o(\lambda r)\, \frac{d\lambda}{\lambda} \tag{5}$$

where J_o is the Bessel function of the first kind of zero order.

The function $A(\lambda)$ can be determined by means of the boundary conditions along $z = 0$. It can be shown that $A(\lambda)$ must satisfy the following triple integrals for the two special cases just given above:

1) Isothermal

$$\int_0^\infty A(\lambda)\, J_o(\lambda r)\, \frac{d\lambda}{\lambda} = T_o, \qquad a < r < b \tag{6a}$$

$$\int_0^\infty A(\lambda)\, J_o(\lambda r)\, d\lambda = 0, \qquad 0 \le r < a \tag{6b}$$

$$\int_0^\infty A(\lambda)\, J_o(\lambda r)\, d\lambda = 0, \qquad b < r \tag{6c}$$

2) Constant Flux

$$\int_o^\infty A(\lambda)\, J_o(\lambda r)\, d\lambda = \frac{Q}{k\pi(b^2 - a^2)}, \quad a < r < b \tag{6d}$$

$$\int A(\lambda)\, J_o(\lambda r)\, d\lambda = 0, \quad 0 \leq r < a \tag{6e}$$

$$\int_o^\infty A(\lambda)\, J_o(\lambda r)\, d\lambda = 0, \quad b < r \tag{6f}$$

In both cases, $A(\lambda)$ must satisfy simultaneously the triple infinite integrals given by Eqs. (6a-6f), respectively. With

$$\overline{T} = \frac{1}{\pi(b^2 - a^2)} \int_a^b T(r,o)\, 2\pi r dr \tag{7}$$

it is easy to show that, with Eq. (5), Eq. (7) becomes, in general,

$$\overline{T} = \frac{2}{(b^2 - a^2)} \int_a^b \int_o^\infty \left[A(\lambda)\, J_o(\lambda r)\, \frac{d\lambda}{\lambda} \right] rdr \tag{8}$$

for the isothermal case $\overline{T} = T_o$. The total heat flow rate can be determined as follows:

$$Q = \int_a^b -k \left.\frac{\partial T}{\partial z}\right|_{z=0} 2\pi r dr \tag{9}$$

Taking the gradient of Eq. (5) with respect to z at z = 0 and substituting it into Eq. (9) yields the general expression for the heat flow rate:

$$Q = 2\pi k \int_z^b \left[\int_o^\infty A(\lambda)\, J_o(\lambda r)\, d\lambda \right] rdr \tag{10}$$

For the constant-flux case, the total heat flow rate is simply Q. Equations (8) and (10) now can be used to obtain the general dimensionless thermal constriction parameter $\psi_a \equiv bkR$:

$$\psi_a = \frac{1}{\pi b(1-\varepsilon^2)} \frac{\int_a^b \left[\int_0^\infty A(\lambda)\ J_o(\lambda r)\ \frac{d\lambda}{\lambda} \right] rdr}{\int_a^b \left[\int_0^\infty A(\lambda)\ J_o(\lambda r)\ d\lambda \right] rdr} \tag{11}$$

where $\varepsilon = a/b < 1$.

The solution of Eq. (11) is dependent upon one's ability to solve the triple infinite integrals for the function $A(\lambda)$. Since the theory of triple infinite integrals has not been developed yet to the point where solutions for Eqs. (6a-6f) (let alone the general cases) are readily obtainable, it is almost impossible to obtain analytic solutions to Eq. (11). Because of these insurmountable difficulties encountered in the classical approach to this problem, it was decided to obtain a solution for the uniform flux case using the integral approach.[4]

Integral Solution

In a recent paper,[4] Yovanovich developed integral expressions for the temperature field and contact area temperature, as well as the thermal constriction resistance for arbitrary flux distributions over arbitrary, planar contact areas supplying heat to insulated half-spaces. In general, the constriction resistance can be evaluated by means of the following expression:

$$R = \frac{1}{2\pi k\Gamma} \frac{\iint_\Gamma \left[\iint_\Gamma \frac{q d\Gamma}{r} \right] d\Gamma}{\iint_\Gamma q d\Gamma} \tag{12}$$

where q is the flux distribution over the contact area Γ, and r is the distance in the contact plane from the elemental heat source $qd\Gamma$ and the point under consideration. The double integral in the numerator represents the evaluation of the local temperature and then the average temperature. The integral in the denominator is the total heat flow rate.

For the case of a circular, annular contact area with uniform heat flux, the contact area temperature is[4]

$$T = \frac{2}{\pi}\frac{qb}{k}\left[E\left(\frac{r}{b}\right) - \left(\frac{r}{b}\right) E\left(\frac{a}{r}\right) + \left(\frac{r}{b}\right)\left\{1 - \left(\frac{a}{r}\right)^2\right\} K\left(\frac{a}{r}\right)\right] \tag{13}$$

where K and E are complete elliptic integrals of the first and second kind, respectively.

If we let $\kappa = a/r$ and, as before, $\varepsilon = a/b$, Eq. (13) can be rewritten as

$$T(\kappa) = \frac{2qb}{\pi k}\left[E\left(\frac{\varepsilon}{\kappa}\right) - \varepsilon\kappa\, K(\kappa) + \frac{\varepsilon}{\kappa} K(\kappa) - \frac{\varepsilon}{\kappa} E(\kappa)\right] \tag{14}$$

After substitution of Eq. (14) into Eq. (7), we obtain for the average contact area temperature the expression

$$\bar{T} = \frac{-2\,\varepsilon^2}{(1-\varepsilon^2)} \int_1^{\varepsilon} \frac{T(\kappa)d\kappa}{\kappa^3} \tag{15}$$

Equation (15) with Eq. (14) reduces to the evaluation of the four integrals given below:

$$\bar{T} = \frac{-4qa}{\pi k}\left(\frac{\varepsilon}{1-\varepsilon^2}\right)\{I_1 - I_2 + I_3 - I_4\} \tag{16}$$

where

$$I_1 = \int_1^{\varepsilon} \frac{E(\varepsilon/\kappa)d\kappa}{\kappa^3} \tag{17}$$

$$I_2 = \varepsilon \int_1^{\varepsilon} \frac{K(\kappa)d\kappa}{\kappa^2} \tag{18}$$

$$I_3 = \varepsilon \int_1^{\varepsilon} \frac{K(\kappa)d\kappa}{\kappa^4} \tag{19}$$

$$I_4 = \varepsilon \int \frac{E(\kappa)d\kappa}{\kappa^4} \tag{20}$$

Solutions to these integrals cannot be found in the handbooks on elliptic functions.[5,6] Our evaluation of these four integrals is presented next.

Integral Evaluation

The integral, Eq. (17), can be transformed into one that is tabulated if we let $\gamma = \varepsilon/\kappa$. Then,

$$d\kappa = -(\varepsilon/\gamma^2)\, d\gamma, \quad \kappa^3 = \varepsilon^3/\gamma^3 \tag{21}$$

With Eq. (21), I_1 becomes

$$I_1 = \frac{1}{\varepsilon^2} \int_1^{\varepsilon} E(\gamma)\, \gamma d\gamma \tag{22}$$

The solution to the integral in Eq. (22) is available[6]:

$$\int_1^{\varepsilon} E(\gamma)\, \gamma d\gamma = \frac{1}{3}\{(1+\varepsilon^2)\, E(\varepsilon) - (1-\varepsilon^2)\, K(\varepsilon) - 2E(1)\} \tag{23}$$

With $E(1) = 1$, Eq. (22) with Eq. (23) yields

$$I_1 = \frac{(1+\varepsilon^2)}{3\varepsilon^2} E(\varepsilon) - \frac{(1-\varepsilon^2)}{3\varepsilon^2} K(\varepsilon) - \frac{2}{3\varepsilon^2} \tag{24}$$

The integral in Eq. (18) is tabulated[6]:

$$\int_1^{\varepsilon} \frac{K(\kappa)\, d\kappa}{\kappa^2} = \left[\frac{E(\kappa)}{\kappa}\right]_1^{\varepsilon} = 1 - \frac{E(\varepsilon)}{\varepsilon} \tag{25}$$

Therefore,

$$I_2 = \varepsilon - E(\varepsilon) \tag{26}$$

The integrals in Eqs. (19) and (20) can be combined to form a new integral I_5:

$$I_5 = I_3 - I_4$$

$$= \varepsilon \int_1^{\varepsilon} [K(\kappa) - E(\kappa)] \frac{d\kappa}{\kappa^4} \tag{27}$$

The solution of Eq. (27) will be effected by using integration by parts where we make the transformation

$$u = K - E, \qquad du = \kappa E(\kappa)\,\frac{d\kappa}{\kappa'^2} \tag{28a}$$

$$dv = \frac{d\kappa}{\kappa^4}, \qquad v = -\frac{1}{3\kappa^3} \tag{28b}$$

where κ' is the complementary modulus with respect to κ.

By means of Eq. (28), Eq. (27) becomes

$$I_5 = uv - \int v\,du$$

$$= -\varepsilon\left[\frac{K(\kappa) - E(\kappa)}{3\kappa^3}\right]_1^{\varepsilon} - \varepsilon\int_1^{\varepsilon} \frac{-1}{3\kappa^3}\,\frac{\kappa E(\kappa)\,d\kappa}{\kappa'^2} \tag{29}$$

After substitution of the limits of integration, Eq. (29) becomes

$$I_5 = \frac{-K(\kappa)}{3\varepsilon^2} + \frac{E(\kappa)}{3\varepsilon^2} + \frac{\varepsilon K(1)}{3} - \frac{\varepsilon}{3} + \frac{\varepsilon}{3}\int_1^{\varepsilon} \frac{E(\kappa)\,d\kappa}{\kappa^2\,\kappa'^2} \tag{30}$$

The remaining integral in Eq. (30) will be evaluated using the identity[6]

$$\int [K(\kappa) - \kappa'^2 K(\kappa)]\,\frac{d\kappa}{\kappa^2\kappa'^2} = \frac{1}{\kappa}\,[K(\kappa) - E(\kappa)] \tag{31}$$

But, we can rewrite Eq. (31):

$$\int \frac{E(\kappa)\,d\kappa}{\kappa^2\,\kappa'^2} - \int \frac{K(\kappa)\,d\kappa}{\kappa^2} = \frac{K(\kappa)}{\kappa} - \frac{E(\kappa)}{\kappa} \tag{32}$$

Recognizing that

$$\int \frac{K(\kappa)\,d\kappa}{\kappa^2} = -\frac{E(\kappa)}{\kappa} \tag{33}$$

we now can put

$$\int^{\varepsilon} \frac{E(\kappa)\,d\kappa}{\kappa^2\,\kappa'^2} = \left[\frac{K(\kappa)}{\kappa} - \frac{2E(\kappa)}{\kappa}\right]_1^{\varepsilon}$$

$$= \frac{K(\varepsilon)}{\varepsilon} - \frac{2E(\varepsilon)}{\varepsilon} - \frac{K(1)}{1} + \frac{2E(1)}{1} \qquad (34)$$

Substitution of Eq. (34) into Eq. (30) gives us

$$I_5 = \frac{E(\varepsilon)}{3\varepsilon^2} - \frac{K(\varepsilon)}{3\varepsilon^2} + \frac{K(\varepsilon)}{3} - \frac{2E(\varepsilon)}{3} + \frac{\varepsilon}{3} \qquad (35)$$

With Eqs. (24, 26, and 35), Eq. (16) becomes, after much simplification,

$$\overline{T} = \frac{8}{3\pi}\frac{qb}{k}\frac{1}{(1-\varepsilon^2)}[1+\varepsilon^3 + (1-\varepsilon^2)\,K(\varepsilon) - (1+\varepsilon^2)\,E(\varepsilon)] \qquad (36)$$

the average contact area temperature as a function of the uniform flux, thermal conductivity, outer radius, and a complex geometric parameter.

Since the total heat flow rate is $Q = q\pi b^2\,(1-\varepsilon^2)$, the dimensionless constriction resistance given by Eq. (12) is found to be

$$R^* = \frac{8}{3\varepsilon^2}\frac{1}{(1-\varepsilon^2)^2}[1+\varepsilon^3 + (1-\varepsilon^2)\,K(\varepsilon) - (1+\varepsilon^2)\,E(\varepsilon)] \qquad (37)$$

where $R^* = kbR = \psi_a$, the constriction parameter of Eq. (11).

It can be seen that Eq. (37) gives $\psi_a(\varepsilon=0) = 8/3\pi^2$, in agreement with the disk solution.[7] Some typical values of ψ_a for several values of ε are presented in Table 1.

It is interesting to note that ψ_a decreases slightly as ε increases from 0 to some value near 0.4; thereafter ψ_a increases monotonically as ε approaches unity. The minimum value of ψ_a is approximately 0.2660. The asymptote of ψ_a as ε tends toward unity was determined by means of a correlation of the semilogarithmic plot of ψ_a against $(\varepsilon^{-1} - 1)$. Utilizing

Table 1 Typical values of ψ_a vs. ε

ε	ψ_a
0.1	0.2695
0.2	0.2680
0.3	0.2667
0.4	0.2660
0.5	0.2666
0.6	0.2691
0.7	0.2746
0.8	0.2858
0.9	0.3109
0.95	0.3390

$$\psi_a = C_1 \ell n[C_2/(\varepsilon^{-1} - 1)] \tag{38}$$

as the correlation equation, it was observed that a very good fit could be obtained if $C_1 = 0.0503$ and $C_2 = 39.66$. It was decided to use a slightly less accurate expression:

$$\psi_a = 0.050 \ \ell n[40/(\varepsilon^{-1} - 1)] \tag{39}$$

Equation (39) differs from the exact expression, Eq. (37), by less than 0.44% when $1/\varepsilon \le 1.005$. At $1/\varepsilon = 1.011$, the error is 1.92%.

Comparison with Results of Smythe[2] and Cooke[3]

Smythe,[2] by a rather ingenious superposition of known solutions, obtained two approximate expressions for the capacitance of an isopotential circular, infinitely thin annulus. Nondimensionalizing Smythe's capacitance expressions allows us to determine the constriction parameter for an isothermal annular contact. These expressions are

$$\psi_a^T = \frac{1}{\pi^2}\left[\frac{\ell n\ 16 + \ell n\ [(1+\varepsilon)/(1-\varepsilon)]}{(1+\varepsilon)}\right] \tag{40}$$

for $1.000 < 1/\varepsilon < 1.100$, and

$$\psi_a^T = \frac{(\pi/8)}{[\cos^{-1}\varepsilon + \sqrt{1-\varepsilon^2}\ \tanh^{-1}\varepsilon][1 + 0.0143\varepsilon^{-1}\ \tan^3(1.28\varepsilon)]} \tag{41}$$

for $1.1 < 1/\varepsilon < \infty$. Typical values of ψ_a^T are presented in Table 2.

Table 2 Smythe's values of ψ_a^T vs. ε

$1/\varepsilon$	ψ_a^T
1.01	0.4113
1.05	0.3365
1.10	0.3085
1.20	0.2855
1.40	0.2683
1.60	0.2610
1.80	0.2570
2.00	0.2548
3.00	0.2510
5.00	0.2501

Table 3 Cooke's values of ψ_a^T vs. ε

$1/\varepsilon$	ψ_a^T
1.021	0.3748
1.091	0.3120
1.125	0.3000
1.20	0.2848
1.25	0.2785
1.50	0.2635
2.00	0.2548

Cooke[3] reported some values of the capacitance predicted by his complex analysis. These values were determined numerically by means of an elaborate and costly computer program, and they are reported in Table 3. It is apparent from an examination of Tables 1-3 that the results of Smythe[2] and Cooke[3] are essentially in agreement. The constant flux constriction parameter (Table 1) is seen to be greater than the isothermal constriction parameter (Table 2) for all values of the geometric parameter ε.

The maximum difference between the constant flux and isothermal cases is about 8.08% when $\varepsilon = 0$. The percent difference decreases as ε increases, becoming less than 1% when $1/\varepsilon = 1.01$. As $1/\varepsilon$ approaches unity, the difference becomes negligibly small; for example, when $1/\varepsilon = 1.0001$, the difference is only 0.434%. Thus we conclude that, for circular, annular contact areas produced by the elastic contact of an O-ring and a smooth flat, the effect of the boundary condition is negligibly small, and one can use either the solution for the isothermal contact or the constant flux to determine the constriction resistance. Since Eq. (39) is equivalent to Eq. (40) but much simpler, it is recommended that one use Eq. (39) for predicting the thermal constriction resistance of a circular, annular contact.

Acknowledgments

The authors gratefully acknowledge the financial support of the National Research Council. One of the authors (GES) acknowledges the support of the Communications Research Centre Department of Communications.

References

[1]Schneider, G.E. and Yovanovich, M.M., "Correlation of the Overall Thermal Resistance of Metallic O-Rings Contacting Two Cylinders," Journal of Spacecraft and Rockets, Vol. 11, March 1974, pp. 206-208.

[2]Smythe, W.R., "The Capacitance of a Circular Annulus," American Journal of Applied Physics, Vol. 22, Dec. 1951, pp. 1499-1501.

[3]Cooke, J.C., "Triple Integral Equations," Quarterly Journal of Mechanics and Applied Mathematics, Vol. 16, May, 1962, pp. 193-203.

[4]Yovanovich, M.M., "Thermal Constriction Resistance of Contacts on a Half-Space: Integral Formulation," AIAA Paper 75-708, Denver, Colo. May 27-29, 1975; also, AIAA Progress in Astronautics and Aeronautics: Radiative Transfer and Thermal Control, Vol. 49, edited by A.M. Smith, New York, 1976, pp. 397-418.

[5]Byrd, P.F. and Friedman, M.D., Handbook of Elliptic Integrals for Engineers and Scientists, Springer-Verlag, New York, 1971.

[6]Gradshteyn, I.S. and Ryzhik, I.M., Table of Integrals, Series and Products, Academic Press, New York, 1965.

[7]Strong, A.B., Schneider, G.E., and Yovanovich, M.M., "Thermal Constriction Resistance of a Disk with Arbitrary Heat Flux - Finite Difference Solution in Oblate Spheroidal Coordinates," also AIAA Progress in Astronautics and Aeronautics: Heat Transfer with Thermal Control Applications, Vol. 39, edited by M.M. Yovanovich, New York, 1975, pp. 65-78.