A GENERAL EXPRESSION FOR PREDICTING CONDUCTION SHAPE FACTORS

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Abstract

The present analytical study describes a method for obtaining conduction shape factors for systems bounded by two isothermal surfaces. Several important cylindrical and rotational systems are examined, and for each system expressions for the shape factors are obtained for the three principal coordinate directions. A knowledge of the metric or Lamé's coefficients of a system leads directly to the determination of the corresponding shape factors. The results of the analysis are applied to several special conduction problems to illustrate the ease of generating shape factors for complex thermal systems.

Nomenclature

a	= focal length of an elliptic cross section
b	= semimajor axis
c	= semiminor axis
C_1	= constant of integration, Eq. (14)
C_2	= constant of integration, Eq. (15)
d	= diameter
g	= metric or Lamé's coefficients, Eq. (2)
k	= thermal conductivity

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k_o	=	thermal conductivity at some reference temperature
L	=	length of the system
q	=	heat flux
Q	=	heat flow rate
R	=	thermal resistance, Eq. (22)
s	=	arc length, Eq. (2)
S	=	surface normal to heat flux vector
T	=	temperature
u_1, u_2, u_3	=	orthogonal curvilinear coordinates
r, ψ, z	=	circular cylinder coordinates
r, θ, ψ	=	spherical coordinates
η, θ, ψ	=	oblate or prolate spheroidal coordinates
η, θ, z	=	elliptic cylindrical coordinates
α	=	thermal coefficient for thermal conductivity, Eq. (21)
β	=	limits on the variable θ
γ	=	limits on the variable ψ
θ	=	transformed temperature, Eq. (9)

Introduction

There is at present no general formula available for predicting stationary conduction shape factors that are needed by today's aerospace and mechanical engineers to analyze complex thermal systems consisting of many components that interface with each other. If the components have simple shapes and heat conduction takes place between isothermal surfaces, then the simple, well-known, conduction shape factors for plates, hollow spheres, and infinitely long hollow cylinders can be used. If, however, the components have complex shapes, as is usually the case, and there is heat conduction between isothermal surfaces, then the thermal analyses become very difficult, and the engineer often must conduct costly thermal tests to determine the shape factors. The other costly alternative is to obtain an approximate solution to the problem by means of a numerical analysis using either a finite-difference approach or a finite element technique. The numerical analyses must be used whenever the thermal problem is posed in the inappropriate coordinate system. For example, if the conduction shape factor for the infinitely long hollow cylinder were required and the analysis were based upon the Cartesian coordinates, then a costly computer program would be required to produce an approximate answer. Obviously, the correct approach is to recognize which coordinate system is appropriate for the geometry under consideration, and then to make a heat balance on the corresponding elemental volume, thus yielding the governing differential equation. Isothermal boundary conditions can then be used to obtain the temperature distribution within the component. The local heat flux can be calculated from the temperature distribution, and the total heat flow rate through the component can then be determined by integration of the product of the local heat flux and corresponding heat flow area over one of the boundaries. The thermal resistance of the component is obtained from the definition of thermal resistance: total temperature drop across the component divided by the total heat flow rate. Each time one encounters a thermal component having a different geometry, the procedure just described must be followed.

The aim of this paper is to develop a general formula for predicting the thermal resistance (conductance shape factor) of thermal components whose physical and geometric characteristics are fully specified. Shape factors for several useful cylindrical and rotational systems will be examined in detail, and it will be shown how these results can be used to resolve some rather complex thermal problems. The results will be applicable to coordinate systems whose solutions are simply separable. The boundary conditions are the simple isothermal type.

Formulation of the Problem in General Coordinates

Consider the steady flow of heat Q from an isothermal surface S_1 at temperature T_1 through a homogeneous medium of thermal conductivity k to a second isothermal surface S_2 at temperature T_2 ($T_1 > T_2$). The stationary temperature field will depend upon the geometry of the isothermal boundary surfaces. When these isothermal surfaces can be made coincident with a coordinate surface by a judicious choice of coordinates, then the temperature field will be one-dimensional in

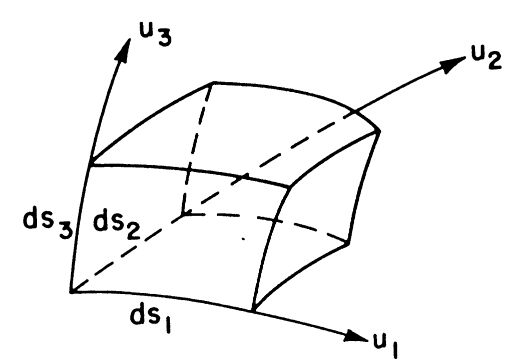

Fig. 1 Orthogonal curvilinear parallelopiped.

that coordinate system. In other words, heat conduction occurs across two surfaces of an orthogonal curvilinear parallelopiped (Fig. 1), whereas the remaining four coordinate surfaces are adiabatic.

Let the general coordinates u_1, u_2, u_3 be so chosen that $T = T(u_1)$ and, therefore, $\partial T/\partial u_2 = \partial T/\partial u_3 = 0$. Under these conditions, the heat flux vector will have one component in the u_1-direction:

$$q_1 = -k(dT/ds) = (-k\ (dT/\sqrt{g_1})/du_1) \qquad (1)$$

where $\sqrt{g_1}$ is the metric or Lamé's coefficient in the u_1-direction. The metric coefficients are defined by the general line element ds expressed in terms of the differentials of arc-lengths on the coordinate lines[1,2]:

$$(ds)^2 = g_1(du_1)^2 + g_2(du_2)^2 + g_3(du_3)^2 \qquad (2)$$

The product terms such as $du_i\ du_j$ $(i \neq j)$ do not appear because of the orthogonality property of the chosen coordinate system. These metric coefficients can also be generated by means of the following formula[1]:

$$g_i \equiv (\partial x/\partial u_i)^2 + (\partial y/\partial u_i)^2 + (\partial z/\partial u_i)^2 \qquad i = 1,2,3$$

provided that the Cartesian coordinates x, y, z can be expressed in terms of the new coordinates u_1, u_2, u_3 by the equations

$$x = x(u_1,u_2,u_3),\ y = y(u_1,u_2,u_3),\ z = z(u_1,u_2,u_3)$$

The elemental coordinate surface located at u_1 and orthogonal to the u_1 direction is therefore

$$dS_1 = ds_2\ ds_3 = \sqrt{g_2\ g_3}\ du_2\ du_3 \qquad (3)$$

and the heat flow per unit time through this surface into the volume element is, according to Fourier,

$$Q_1 = -k\ dS_1\ (dT/ds_1) = -k(\sqrt{g}/g_1)\ (dT/du_1)\ du_2\ du_3 \qquad (4)$$

where $g \equiv g_1\ g_2\ g_3$. The heat flow rate out of the volume element will, therefore, be

$$Q_1 + (dQ_1/ds_1)\ ds_1 = Q_1 + (dQ_1/du_1)\ du_1 \qquad (5)$$

The net rate of heat conduction out of the volume element in the u_1 direction is, accordingly,

$$(d/du_1)\ [k(\sqrt{g}/g_1)\ (dT/du_1)]\ du_1\ du_2\ du_3 \qquad (6)$$

For steady-state conditions with no heat sources within the element, Laplace's equation in general coordinates is obtained by dividing Eq. (6) by the elemental volume $\sqrt{g}\ du_1\ du_2\ du_3$ and equating to zero:

$$(1/\sqrt{g})\ (d/du_1)\ [k(\sqrt{g}/g_1)\ (dT/du_1)] = 0 \qquad (7)$$

Equation (7) is the governing differential equation and is nonlinear when k is not a constant.

The uniform temperature boundary conditions are

$$\begin{aligned} u_1 &= a \qquad T = T_1 \\ u_1 &= b \qquad T = T_2 \end{aligned} \qquad (8)$$

Equation (7) can, however, be reduced to a linear differential equation by introducing a new temperature θ related to the temperature T by the Kirchhoff transformation,[3]

$$\theta = (1/k_o)\int_o^T k\ dT \qquad (9)$$

where k_o denotes the value of the thermal conductivity at some convenient reference temperature, say T = 0. It follows from Eq. (9) that

$$d\theta/du_1 = (k/k_o)\ (dT/du_1) \qquad (10)$$

and, therefore, Eq. (7) reduces to

$$(d/du_1)\ [(\sqrt{g}/g_1)\ (d\theta/du_1)] = 0 \qquad (11)$$

after multiplying through by $\sqrt{g}/k_o$. The boundary conditions, according to Eq. (9), now become

$$\begin{aligned} u_1 &= a \qquad \theta = \theta_1 = (1/k_o)\int_o^{T_1} k\ dT \\ u_1 &= b \qquad \theta = \theta_2 = (1/k_o)\int_o^{T_2} k\ dT \end{aligned} \qquad (12)$$

The solution of Eq. (11) is

$$\theta = C_1\int_o^{u_1} (g_1/\sqrt{g})\ du_1 + C_2 \qquad (13)$$

where C_1 and C_2 are constants of integration. It can readily be shown that

$$C_1 = -(\theta_1 - \theta_2) \Big/ \int_a^b (g_1/\sqrt{g})\, du_1 \tag{14}$$

and

$$C_2 = \theta_1 - C_1 \int_o^a (g_1/\sqrt{g})\, du_1 \tag{15}$$

The temperature distribution is obtained by substitution of Eqs. (14) and (15) into Eq. (13), yielding

$$\frac{\theta_1 - \theta}{\theta_1 - \theta_2} = \frac{\int_a^{u_1} (g_1/\sqrt{g})\, du_1}{\int_a^b (g_1/\sqrt{g})\, du_1}, \quad a < u_1 < b \tag{16}$$

The local heat flux in the u_1 direction [Eq. (1)] becomes, by means of Eqs. (9) and (16),

$$q_1 = \frac{-k_o}{\sqrt{g_1}} \frac{d\theta}{du_1} = \frac{k_o(\theta_1 - \theta_2)}{\sqrt{g_2\, g_3} \int_a^b (g_1/\sqrt{g})\, du_1} \tag{17}$$

The heat flow rate through the elemental surface dS_1 is, therefore,

$$q_1\, dS_1 = \frac{k_o(\theta_1 - \theta_2)}{\int_a^b (g_1/\sqrt{g})\, du_1}\, du_2\, du_3 \tag{18}$$

The total heat flow rate through the thermal system can be obtained by integration of Eq. (18) between the appropriate limits. Therefore,

$$Q = k_o\, (\theta_1 - \theta_2) \int_{u_2} \int_{u_3} \frac{du_2\, du_3}{\int_a^b (g_1/\sqrt{g})\, du_1} \tag{19}$$

An examination of Eq. (9) shows that

$$k_o(\theta_1 - \theta_2) = \int_{T_2}^{T_1} k\, dT = k_a\, (T_1 - T_2) \tag{20}$$

where k_a is the average value of the thermal conductivity and is given by

$$k_o\, [1 + (\alpha/2)\, (T_1 + T_2)] \tag{21}$$

if $k = k_o(1 + \alpha T)$.

The definition of thermal resistance of a system (total temperature drop across the system divided by the total heat flow rate) yields the following general expression for the conduction shape factor:

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$$S_1 = (R\, k_a)^{-1} = \int_{u_2} \int_{u_3} \frac{du_2\, du_3}{\int_a^b (g_1/\sqrt{g})\, du_1} \tag{22}$$

The right-hand side of Eq. (22) has the dimensions of length and is therefore purely geometric. There are several very important and useful coordinate systems that can be used to solve many seemingly complex conduction problems. Since each coordinate system has three principal directions, there are three sets of shape factors corresponding to each of these directions. The conduction shape factors for several coordinate systems are given in the following section. This section by no means represents the total number of coordinate systems which are amenable to this type of analysis. It does, however, contain the most frequently used coordinate systems.

General Expressions for Conduction Shape Factors

1. Circular Cylinder Coordinates: (r,ψ,z), Fig. 2

$$(ds)^2 = (dr)^2 + r^2\, (d\psi)^2 + (dz)^2$$

$$g_r = 1,\; g_\psi = r^2,\; g_z = 1,\; \sqrt{g} = r$$

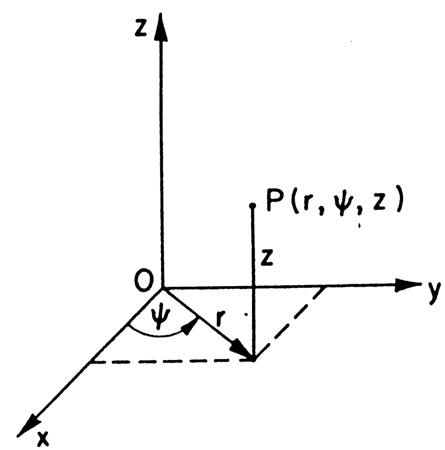


Fig. 2 Circular cylinder coordinates.

a) r Direction. Let $u_1 = r$; therefore, $u_2 = \psi$, $u_3 = z$, and $g_1/\sqrt{g} = 1/r$. If

$$a < r < b$$
$$0 < \psi < \beta \qquad \beta_{max} = 2\pi$$
$$0 < z < L$$

then

$$R\,k_a = \frac{\ln(b/a)}{\beta L} \tag{23}$$

b) ψ Direction. Let $u_1 = \psi$; therefore, $u_2 = z$, $u_3 = r$, and $g_1/\sqrt{g} = r$. Limits of integration were given previously:

$$R\,k_a = \frac{\beta}{L\,\ln(b/a)} \tag{24}$$

c) z Direction. Let $u_1 = z$; therefore, $u_2 = r$, $u_3 = \psi$, and $g_1/\sqrt{g} = 1/r$. Limits of integration were given previously:

$$R\,k_a = \frac{2L}{\beta\,(b^2 - a^2)\,L} \tag{25}$$

2. Spherical Coordinates: (r,θ,ψ), Fig. 3

$$(ds)^2 = (dr)^2 + r^2\,(d\theta)^2 + r^2\,\sin^2\theta\,(d\psi)^2$$
$$g_r = 1,\ g_\theta = r^2,\ g_\psi = r^2\,\sin^2\theta,\ \sqrt{g} = r^2\,\sin\theta$$

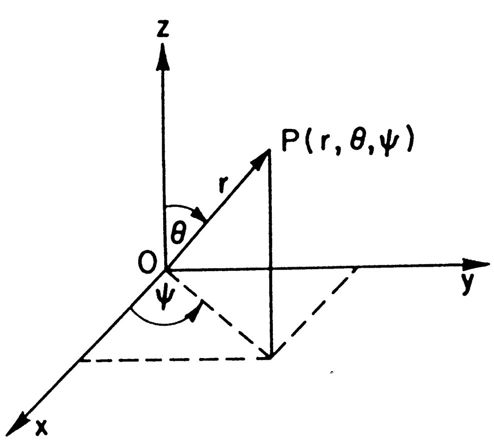


Fig. 3 Spherical coordinates.

a) r Direction. Let $u_1 = r$; therefore, $u_2 = \theta$, $u_3 = \psi$, and $g_1/\sqrt{g} = 1/(r^2\,\sin\theta)$. If

$$a < r < b$$
$$\beta_1 < \theta < \beta_2 \qquad \beta_{min} = 0 \text{ and } \beta_{max} = \pi$$
$$0 < \psi < \gamma \qquad \gamma_{max} = 2\pi$$

then

$$R\,k_a = [(1/a)-(1/b)]/\gamma\,(\cos\beta_1 - \cos\beta_2) \tag{26}$$

b) θ Direction. Let $u_1 = \theta$; therefore, $u_2 = \psi$, $u_3 = r$, and $g_1/\sqrt{g} = 1/(\sin\theta)$. Limits of integration were given previously:

$$R\,k_a = \frac{\ln[\tan(\beta_2/2)] - \ln[\tan(\beta_1/2)]}{\gamma(b - a)} \tag{27}$$

c) ψ Direction. Let $u_1 = \psi$; therefore, $u_2 = r$, $u_3 = \theta$, and $g_1/\sqrt{g} = \sin\theta$. Limits of integration were given previously:

$$R\,k_a = \gamma/(b - a)\{\ln[\tan(\beta_2/2)] - \ln[\tan(\beta_1/2)]\} \tag{28}$$

3. Elliptic Cylinder Coordinates: (η,ψ,z), Fig. 4

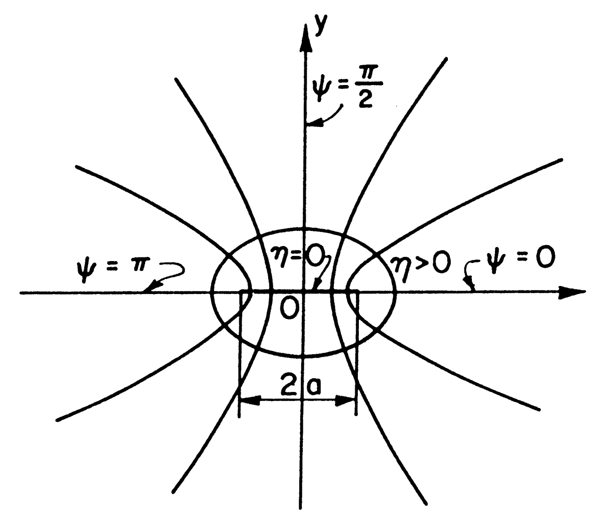


Fig. 4 Elliptic cylinder coordinates.

$$(ds)^2 = a^2(\cosh^2\eta - \cos^2\psi)[(d\eta)^2 + (d\psi)^2] + (dz)^2$$

$$g_\eta = g_\psi = a^2(\cosh^2\eta - \cos^2\psi), \; g_z = 1,$$

$$\sqrt{g} = a^2(\cosh^2\eta - \cos^2\psi)$$

a) η Direction. Let $u_1 = \eta$; therefore, $u_2 = \psi$, $u_3 = z$, and $g_1/\sqrt{g} = 1$. If

$$\eta_1 < \eta < \eta_2 \qquad \eta_{min} = 0, \; \eta_{max} = \infty$$
$$0 < \psi < \beta \qquad \beta_{max} = 2\pi$$
$$0 < z < L$$

then

$$R\,k_a = (\eta_2 - \eta_1)/L\beta \tag{29}$$

where

$$\eta_1 = \frac{1}{2}\,\ell n\left[\frac{b_1 + c_1}{b_1 - c_1}\right], \; \eta_2 = \frac{1}{2}\,\ell n\left[\frac{b_2 + c_2}{b_2 - c_2}\right]$$

and

$$a = \sqrt{b_1^2 - c_1^2} = \sqrt{b_2^2 - c_2^2}$$

b) ψ Direction. Let $u_1 = \psi$; therefore, $u_2 = z$, $u_3 = \eta$, and $g_1/\sqrt{g} = 1$. Limits of integration were given previously:

$$R\,k_a = \beta/L\,(\eta_2 - \eta_1) \tag{30}$$

c) z Direction. Let $u_1 = z$; therefore, $u_2 = r$, $u_3 = \psi$, and

$$\frac{g_1}{\sqrt{g}} = \frac{1}{a^2(\cosh^2\eta - \cos^2\psi)}$$

Limits of integration were given previously:

$$R\,k_a = \frac{L}{a^2\int_0^\beta\int_{\eta_1}^{\eta_2}(\cosh^2\eta - \cos^2\psi)\,d\eta\,d\psi} \tag{31}$$

4. Bicylinder Coordinates: (η,ψ,z), Fig. 5

$$(ds)^2 = \frac{a^2}{(\cosh\eta - \cos\psi)^2}[(d\eta)^2 + (d\psi)^2] + (dz)^2$$

$$g_\eta = g_\psi = \frac{a^2}{(\cosh\eta - \cos\psi)^2}, \; g_z = 1,$$

$$\sqrt{g} = \frac{a^2}{(\cosh\eta - \cos\psi)^2}$$

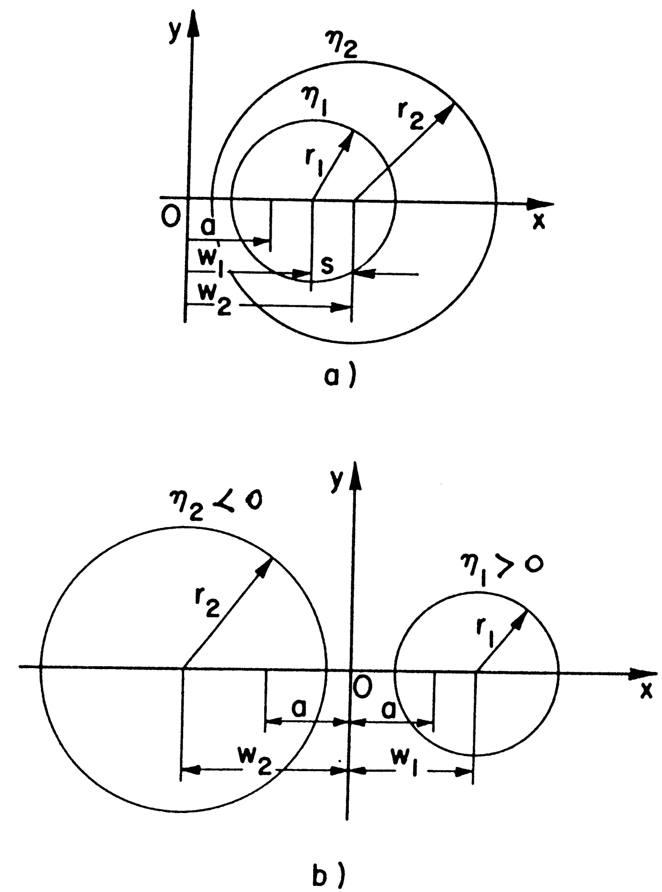


Fig. 5 Bicylinder coordinates.

a) η Direction. Let $u_1 = \eta$; therefore, $u_2 = \psi$, $u_3 = z$, and $g_1/\sqrt{g} = 1$. If

$$\eta_1 > \eta > \eta_2 \qquad \eta_{min} = -\infty,\ \eta_{max} = +\infty$$
$$0 < \psi < \beta \qquad \beta_{max} = 2\pi$$
$$0 < z < L$$

then

$$R\, k_a = (\eta_1 - \eta_2)/\ L\,\beta \tag{32}$$

where

$$|\eta_1| = \sinh^{-1}\sqrt{\left(\frac{w_1}{r_1}\right)^2 - 1}$$
$$|\eta_2| = \sinh^{-1}\sqrt{\left(\frac{w_2}{r_2}\right)^2 - 1}$$

b) ψ Direction. Let $u_1 = \psi$; therefore, $u_2 = z$, $u_3 = \eta$, and $g_1/\sqrt{g} = 1$. Limits of integration were given previously:

$$R\, k_a = \beta/L\, (\eta_1 - \eta_2) \tag{33}$$

c) z Direction. Let $u_1 = z$; therefore, $u_2 = \eta$, $u_3 = \psi$, and

$$\frac{g_1}{\sqrt{g}} = \frac{(\cosh\eta - \cos\psi)^2}{a^2}$$

Limits of integration were given previously:

$$R\, k_a = \frac{L}{a^2\int_o^\beta\int_{\eta_2}^{\eta_1}\frac{d\eta d\psi}{(\cosh\eta - \cos\psi)^2}} \tag{34}$$

where $a = \sqrt{w_1^2 - r_1^2} = \sqrt{w_2^2 - r_2^2}$

5. Oblate Spheroidal Coordinates: (η,θ,ψ), Fig. 6

$$(ds)^2 = a^2(\cosh^2\eta - \sin^2\theta)[(d\eta)^2 + (d\theta)^2]$$
$$+ a^2\cosh^2\eta\, \sin^2\theta\, (d\psi)^2$$
$$g_\eta = g_\theta = a^2(\cosh^2\eta - \sin^2\theta)$$

$$g_\psi = a^2\cosh^2\eta\, \sin^2\theta$$
$$\sqrt{g} = a^3(\cosh^2\eta - \sin^2\theta)\cosh\eta\, \sin\theta$$

a) η Direction. Let $u_1 = \eta$; therefore, $u_2 = \theta$, $u_3 = \psi$, and

$$\frac{g_1}{\sqrt{g}} = \frac{1}{a\cosh\eta\,\sin\theta}$$

If

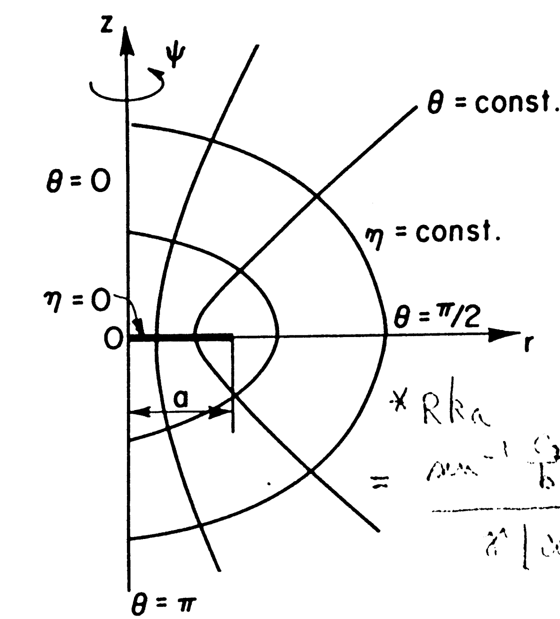
$$\eta_1 < \eta < \eta_2 \qquad \eta_{min} = 0,\ \eta_{max} = \infty$$
$$\beta_1 < \theta < \beta_2 \qquad \beta_{min} = 0,\ \beta_{max} = \pi$$
$$0 < \psi < \gamma \qquad \gamma_{max} = 2\pi$$

then

$$*R\, k_a = \frac{\tan^{-1}(\sinh\eta_2) - \tan^{-1}(\sinh\eta_1)}{a\,\gamma\,[\cos\beta_1 - \cos\beta_2]} \tag{35}$$

where $a = \sqrt{b_1^2 - c_1^2} = \sqrt{b_2^2 - c_2^2}$

and $\eta_1 = \tanh^{-1}(c_1/b_1)$, $\eta_2 = \tanh^{-1}(c_2/b_2)$



$$*R k_a = \frac{\sin^{-1}\frac{c_2}{b_2} - \sin^{-1}\frac{c_1}{b_1}}{\gamma\,[\cos\beta_1 - \cos\beta_2]}$$

Fig. 6 Oblate spheroidal coordinates.

b) θ Direction. Let $u_1 = \theta$; therefore, $u_2 = \psi$, $u_3 = \eta$, and

$$\frac{g_1}{\sqrt{g}} = \frac{1}{a\cosh\eta\,\sin\theta}$$

Limits of integration were given previously:

$$R\,k_a = \frac{\ell n[\tan(\beta_2/2)] - \ell n[\tan(\beta_1/2)]}{a\,\gamma\,(\sinh\eta_2 - \sinh\eta_1)} \tag{36}$$

c) ψ Direction. Let $u_1 = \psi$; therefore, $u_2 = \eta$, $u_3 = \theta$, and

$$\frac{g_1}{\sqrt{g}} = \frac{\cosh\eta\,\sin\theta}{a\,(\cosh^2\eta - \sin^2\theta)}$$

Limits of integration were given previously:

$$R\,k_a = \frac{\gamma/a}{\int_{\eta_1}^{\eta_2}\int_{\beta_1}^{\beta_2} \frac{(\cosh^2\eta - \sin^2\theta)\,d\theta\,d\eta}{\cosh\eta\,\sin\theta}} \tag{37}$$

6. Prolate Spheroidal Coordinates: (η,θ,ψ), Fig. 7

$$(ds)^2 = a^2(\sinh^2\eta + \sin^2\theta)[(d\eta)^2 + (d\theta)^2]$$
$$+ a^2\sinh^2\eta\,\sin^2\theta\,(d\psi)^2$$
$$g_\eta = g_\theta = a^2(\sinh^2\eta + \sin^2\theta)$$
$$g_\psi = a^2\sinh^2\eta\,\sin^2\theta$$
$$\sqrt{g} = a^3(\sinh^2\eta + \sin^2\theta)\sinh\eta\,\sin\theta$$

a) η Direction. Let $u_1 = \eta$; therefore, $u_2 = \theta$, $u_3 = \psi$, and

$$\frac{g_1}{\sqrt{g}} = \frac{1}{a\sinh\eta\,\sin\theta}$$

If

$\eta_1 < \eta < \eta_2$	$\eta_{min} = 0,$	$\eta_{max} = \infty$
$\beta_1 < \theta < \beta_2$	$\beta_{min} = 0,$	$\beta_{max} = \pi$
$0 < \psi < \gamma$	$\gamma_{max} = 2\pi$	

then

$$R\,k_a = \frac{\ell n[\tanh(\eta_2/2)] - \ell n[\tanh(\eta_1/2)]}{a\,\gamma(\cos\beta_1 - \cos\beta_2)} \tag{38}$$

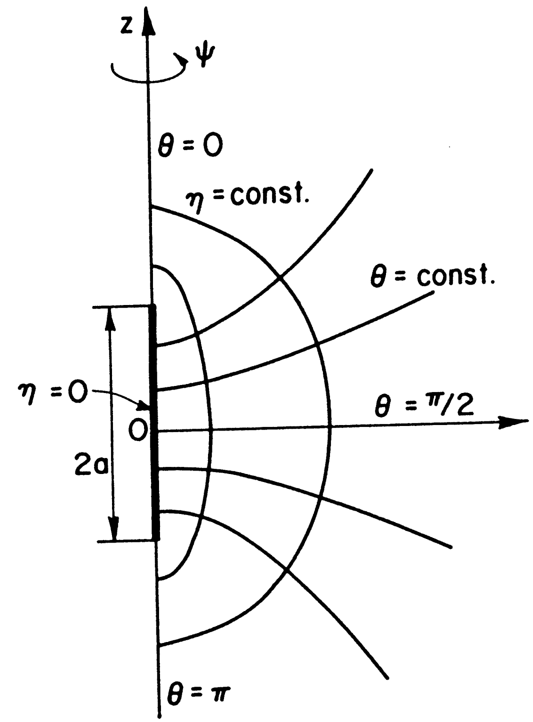


Fig. 7 Prolate spheroidal coordinates.

where

$$a = \sqrt{b_1^2 - c_1^2} = \sqrt{b_2^2 - c_2^2}$$

and

$$\eta_1 = \frac{1}{2}\,\ell n\left[\frac{b_1 + c_1}{b_1 - c_1}\right], \quad \eta_2 = \frac{1}{2}\,\ell n\left[\frac{b_2 + c_2}{b_2 - c_2}\right]$$

b) θ Direction. Let $u_1 = \theta$; therefore, $u_2 = \psi$, $u_3 = \eta$, and

$$\frac{g_1}{\sqrt{g}} = \frac{1}{a\sinh\eta\,\sin\theta}$$

Limits of integration were given previously:

$$R\,k = \frac{\ell n[\tan(\beta_2/2)] - \ell n[\tan(\beta_1/2)]}{\cosh\eta_2 - \cosh\eta_1)} \tag{39}$$

c) ψ Direction. Let $u_1 = \psi$; therefore, $u_2 = \eta$, $u_3 = \theta$, and

$$\frac{g_1}{\sqrt{g}} = \frac{\sinh\eta \sin\theta}{a\,(\sinh^2\eta + \sin^2\theta)}$$

Limits of integration were given previously:

$$R\,k_a = \frac{\gamma/a}{\int_{\eta_1}^{\eta_2}\int_{\beta_1}^{\beta_2} \frac{(\sinh^2\eta + \sin^2\theta)\,d\theta d\eta}{\sinh\eta \sin\theta}} \tag{40}$$

Application of Results to Special Conduction Problems

This part of the paper will be devoted to the application of Eq. (22) and its results, as shown in the previous section, to several conduction problems.

1. Spherical Coordinates

Consider the resistance to steady heat flow in the wall of a hollow sphere of radii a, b, thermal conductivity k (Fig. 8). The isothermal boundaries are $\theta = \beta$ and $\theta = (\pi - \beta)$, $a < r < b$. The surfaces $r = a$ and $r = b$ are perfectly insulated. In this problem, the temperature field is axisymmetric, depending only upon the coordinate θ, and therefore the resistance is given by Eq. (27):

$$R = \frac{1}{2\pi kt}\,\ln\left[\frac{1}{\tan^2(\beta/2)}\right] \tag{41}$$

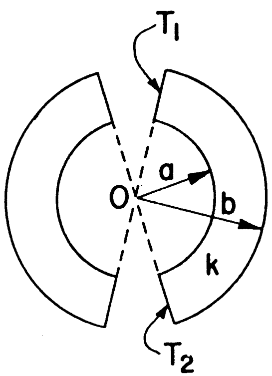


Fig. 8 Spherical problem.

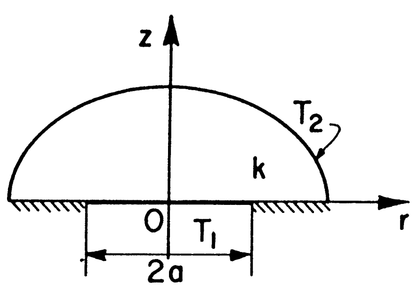


Fig. 9 Elliptical problem.

where $t = (b - a)$ is the thickness of the wall. This expression has been used to predict the thermal resistance through a portion of the wall of the hollow microspheres used as superinsulation[4,5].

2. Elliptic Cylinder Coordinates

In Fig. 9, we see a schematic representing the steady heat flow from an isothermal strip ($\eta = 0$) of width 2a through a medium of conductivity k to a second isothermal surface $\eta = \eta_1$. The surface $y = 0$, $a < x < b$, is perfectly insulated. In this problem, the heat flows in the η direction only, and the thermal resistance is given by Eq. (29):

$$R = \frac{1}{2\pi Lk}\,\ln\left[\frac{b + c}{b - c}\right] \tag{42}$$

where $\beta = \pi$ and L is the length of the system, assumed to be very long. Equation (42) was used by Yovanovich and Coutanceau[6] to obtain a closed-form solution for the thermal

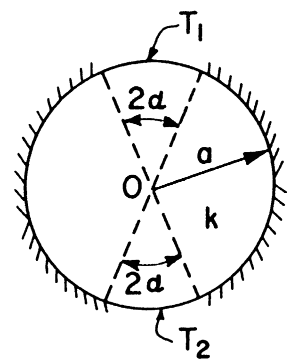


Fig. 10 Conduction across a circular cylinder.

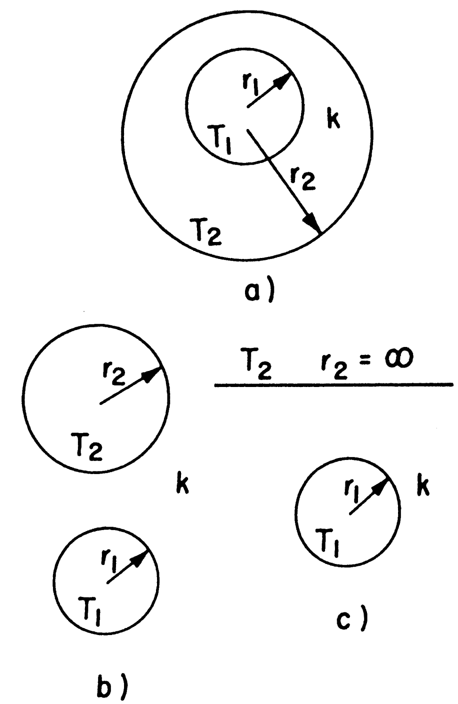


Fig. 11 Bicylindrical problem.

resistance of the system shown in Fig. 10. Equation (42) also can be used to predict the thermal resistance, to less than 1% error, between an isothermal strip of width 2a and a semicircular surface radius b, provided that $a/b < 0.7$. Yovanovich[7,8] recently applied these results to two interesting thermal contact resistance problems.

3. Bicylinder Coordinates

This coordinate system can be used to determine the thermal resistance between eccentric isothermal cylinders (Fig. 11). The one cylinder can be wholly within the second (Fig. 11a) or wholly outside (Fig. 11b). It may be used to predict the resistance of the medium surrounding a buried isothermal pipe loosing heat to an isothermal level surface (Fig. 11c). In all three cases, if the cylinders are assumed to be very long of length L, the heat flows only in the η direction, and the resistances are given by Eq. (32):

$$R = \frac{1}{2\pi kL}\left[\sinh^{-1}\sqrt{\left(\frac{w_1}{r_1}\right)^2 - 1} \pm \sinh^{-1}\sqrt{\left(\frac{w_2}{r_2}\right)^2 - 1}\right] \quad (43)$$

and

$$R = \frac{1}{2\pi kL}\,\ln\left[\frac{w}{r} + \sqrt{\left(\frac{w}{r}\right)^2 - 1}\right] \quad (44)$$

where the negative sign in Eq. (43) is used for Fig. 11a, and the plus sign is used for Fig. 11b. Equation (44) is the thermal resistance for the buried pipe problem.

If, for example, there is heat conduction in the ψ direction as shown in Fig. 12, the exact closed-form expression for the resistance can be written down directly,

$$R = \frac{\pi}{kL}\left\{\cosh^{-1}\frac{r_2^2 - r_1^2 - s^2}{2\,r_1 s} - \cosh^{-1}\frac{r_2^2 - r_1^2 + s^2}{2\,r_2 s}\right\} \quad (45)$$

without having to resort to a complex and costly computer program.

4. Oblate Spheroidal Coordinates

Consider the thermal resistance to steady heat conduction from an isothermal circular disk ($\eta = 0$) of radius a through a medium of conductivity k to an isothermal surface described by an oblate spheroid ($\eta > 0$). The remainder of the surface $z = 0$ lying between $a < r < b$ is perfectly insulated (Fig. 13). In this problem, the heat flows axisymmetrically, in the η

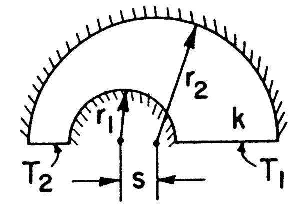


Fig. 12 Bicylindrical problem.

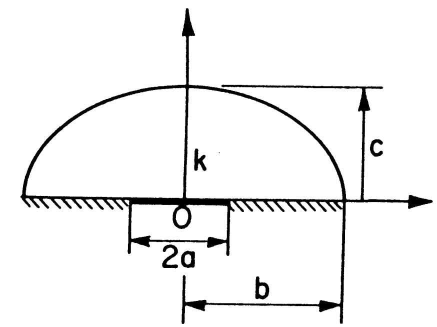


Fig. 13 Oblate spheroidal problem.

direction only, and so the resistance is given by Eq. (35):

$$R = \frac{1}{2\pi ka}\tan^{-1}(\sinh\eta) = \frac{1}{2\pi ka}\tan^{-1}\left[\sinh\left\{\tanh^{-1}\frac{c}{b}\right\}\right] \quad (46)$$

As η becomes very large, $c/b \to 1$, and Eq. (46) reduces to

$$R = 1/4ka \quad (47)$$

the well-known expression[9] for the thermal constriction resistance caused by an isolated isothermal disk situated on an insulated half-space. Yovanovich et al.[10] also used this coordinate system and Eq. (46) to show that 98.4% of the constriction resistance occurs in a volume $r/a = 40$, and practically all or 99.5% occurs in a volume $r/a = 80$.

This coordinate system obviously can give directly the thermal resistance of the medium placed between and in perfect contact with two isothermal oblate spheroidal surfaces η_1 and η_2 (Fig. 14). Using Eq. (35), putting $\gamma = 2\pi$, $\beta_1 = 0$, and $\beta_2 = \pi$, we have

$$R = \frac{1}{4\pi k\sqrt{b_1^2 - c_1^2}}\left\{\tan^{-1}\left[\sinh\left(\tanh^{-1}\frac{c_2}{b_2}\right)\right] - \tan^{-1}\left[\sinh\left(\tanh^{-1}\frac{c_1}{b_1}\right)\right]\right\} \quad (48)$$

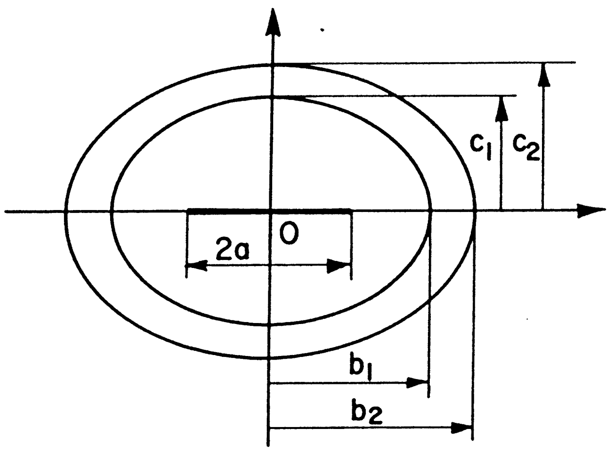


Fig. 14 Oblate spheroidal problem.

For pure conduction from an isothermal oblate spheroid η_1 into an infinite medium, the resistance, according to Eq. (48), is

$$R = \frac{1}{4\pi k\sqrt{b_1^2 - c_1^2}}\left\{\frac{\pi}{2} - \tan^{-1}\left[\sinh\left(\tanh^{-1}\frac{c_1}{b_1}\right)\right]\right\} \tag{49}$$

This coordinate system also can be used to predict the thermal resistance between an isothermal oblate spheroid (η_1) or an isothermal circular disk of radius a ($\eta_1 = 0$) and an isothermal spherical surface (Fig. 15), with less than 1% error, provided that $\eta_2 > 2.4$.

5. Prolate Spheroidal Coordinates

The thermal resistance between two isothermal prolate spheroids where the temperature field depends only upon the η direction (Fig. 16) is given by Eq. (36):

$$R = \frac{1}{4\pi k\sqrt{b_1^2 - c_1^2}}\left\{\ell n\left[\tanh\left(\frac{1}{4}\ell n\frac{b_1 + c_1}{b_1 - c_1}\right)\right] \pm \ell n\left[\tanh\left(\frac{1}{4}\ell n\frac{b_2 + c_2}{b_2 - c_2}\right)\right]\right\} \tag{50}$$

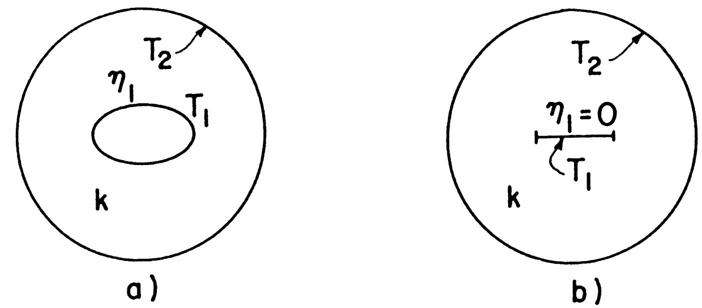


Fig. 15 Oblate-sphere and disk-sphere problem.

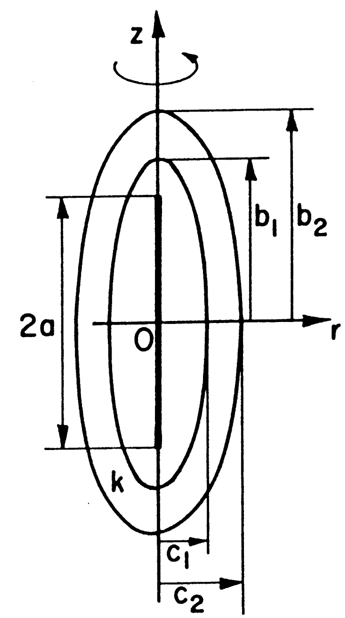


Fig. 16 Prolate spheroidal problem.

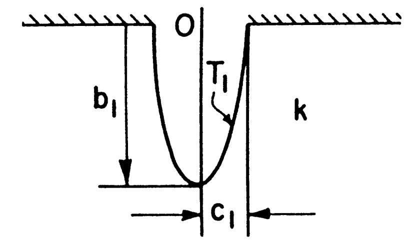


Fig. 17 Prolate spheroidal coordinates.

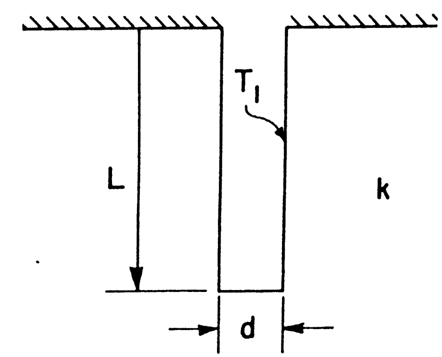


Fig. 18 Prolate spheroidal problem.

where $\gamma = 2\pi$, $\beta_1 = 0$, and $\beta_2 = \pi$. The thermal resistance for pure conduction from an isothermal prolate spheroid into an infinite medium is, according to Eq. (50),

$$R = \frac{1}{4\pi k\sqrt{b_1^2 - c_1^2}}\, \ell n\left[\tanh\left(\frac{1}{4}\, \ell n\, \frac{b_1 + c_1}{b_1 - c_1}\right)\right]^{-1} \tag{51}$$

Equation (51) can be used to predict the resistance of the thermal problem shown in Fig. 17. Here $\beta_1 = \pi/2$, $\beta_2 = \pi$, and $\eta_2 = \infty$, and the resistance is

$$R = \frac{1}{2\pi k\sqrt{b_1^2 - c_1^2}}\, \ell n\left[\tanh\left(\frac{1}{4}\, \ell n\, \frac{b_1 + c_1}{b_1 - c_1}\right)\right]^{-1} \tag{52}$$

For large b_1/c_1, the prolate spheroid can be used to approximate the shape of a circular rod. Since no solution for the thermal resistance of a long cylinder of finite length L and nonvanishing diameters d is known, one can use, to a first approximation, Eq. (52). Letting $2b_1 = L$, $2c_1 = d_1$, it is evident that, for $d/L < 0.1$, the following equation is valid:

$$b_1^2 - c_1^2 \approx b_1\,[1 - (1/2)\,(c_1/b_1)^2] \tag{53}$$

Introducing this in Eq. (52) gives the resistance of an isothermal cylinder, $\eta << 1$ (Fig. 18), as

$$R = (1/2\pi kL)\; \ell n(4L/d) \tag{54}$$

These are but a few selected examples of the use of Eq. (22) and its results as presented in the section entitled "General Expressions for Conduction Shape Factors." Two coordinate systems that have not been discussed will now be presented to show the generality of Eq. (22). Yovanovich[11] showed that the thermal constriction resistance due to an isothermal elliptic contact area supplying heat to a half-space (Fig. 19) is given by

$$R = \psi/4ka \tag{55}$$

where $\psi = 2K/\pi$, and K is the complete integral of the first kind of modulus $(1 - b^2/a^2)$, where a and b are the semimajor and semiminor axes of the contact area. For $a/b > 2$, $\psi = (2/\pi)\; \ell n(4a/b)$, with an error of less than 1%.

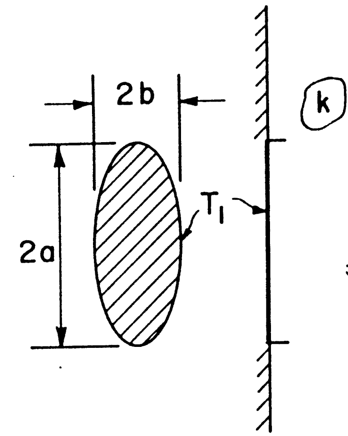


Fig. 19 Elliptical disk problem.

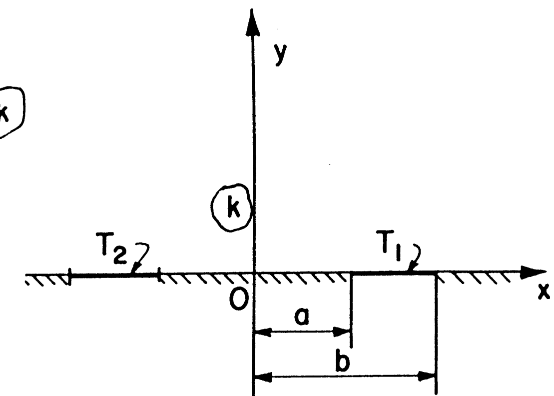


Fig. 20 Coplanar isothermal strips.

Yovanovich showed that this very complex thermal problem, if posed in any coordinate system other than the appropriate one, could not be solved easily.

The last special coordinate system to be discussed here is the system based upon the transformation[12]

$$x + i\,y = a\,\mathrm{sn}(u + iv) \tag{56}$$

and is appropriate to the thermal problem shown in Fig. 20 and specified by the following equations:

$$(\partial^2T/\partial x^2) + (\partial^2T/\partial y^2) = 0 \tag{57}$$

$$\left.\begin{array}{lll} T = T_1, & a < x < b, & y = 0 \\ T = T_2, & -b < x < -a, & y = 0 \\ \dfrac{\partial T}{\partial y} = 0, & 0 < |x| < a,\ |x| > b, & y = 0 \end{array}\right\} \tag{58}$$

Also, T should tend to zero as $\sqrt{x^2 + y^2} \to \infty$. The solution of Eq. (57), subject to the mixed boundary conditions, Eqs. (58), can be written in the form[13]

$$T = \int_0^\infty A(\lambda)\, e^{-\lambda y}\, \sin(\lambda x)\, \frac{d\lambda}{\lambda} \tag{59}$$

where the function $A(\lambda)$ must satisfy the triple integral equations along $y = 0$:

$$\left.\begin{aligned} &\int_0^\infty A(\lambda)\,\sin(\lambda x)\,d\lambda = 0, && 0 < x < a \\ &\int_0^\infty A(\lambda)\,\sin(\lambda x)\,\frac{d\lambda}{\lambda} = T_1, && a < x < b \\ &\int_0^\infty A(\lambda)\,\sin(\lambda x)\,d\lambda = 0, && x > b \end{aligned}\right\} \quad (60)$$

Obviously, the problem is very complex, and it would be extremely difficult to determine the thermal resistance. If, however, the metric coefficients are available, then Eq. (22) can be used to determine the resistance directly without having to solve for the temperature distribution. Moon and Spencer[12] give the following expressions for the metric coefficients:

$$g_\mu = g_\nu = (a^2\,\Omega^2/\Lambda^2),\quad g_z = 1 \qquad (61)$$

where

$$\Lambda = 1 - dn^2\mu\ sn^2\nu \qquad (62)$$

and

$$\Omega^2 = (1 - sn^2\mu\ dn^2\nu)(dn^2\nu - \kappa^2\ sn^2\mu) \qquad (63)$$

and $\kappa = a/b$. The orthogonal coordinates[14] are μ, ν, z, where $\mu = K$ on the isothermal strip $a < x < b$, $\mu = -K$ on the other isothermal strip $-b < x < -a$, and K is the complete elliptic integral of the first kind of modulus a/b. The heat flow lines ν = const are orthogonal to the isotherms, and ν ranges from 0 when $y = 0$, $-a < x < a$, to K' when $y = 0$, $x > b$, and $x < -b$. Here K' is the complete elliptic integral of the first kind of modulus $1 - (a/b)^2$. For the problem shown in Fig. 20, the heat conduction is in the μ direction only, i.e., $T = T(\mu)$ only. Therefore, $g_1/\sqrt{g} = 1$, and Eq. (22),

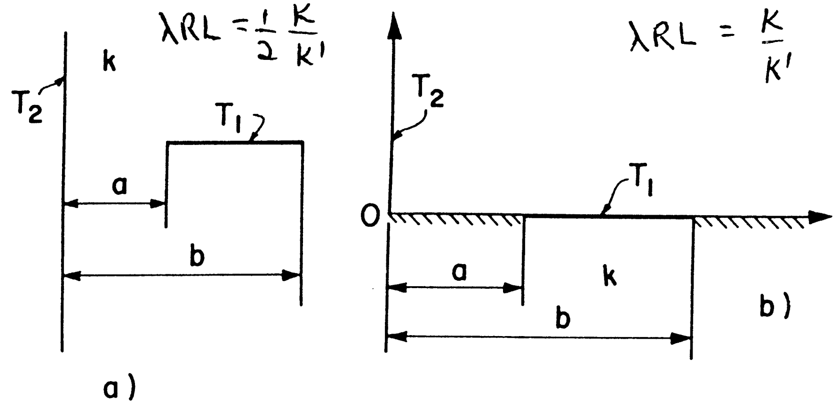


Fig. 21 Coplanar strip problem.

with the appropriate limits of integration, yields

$$R = \frac{K(a/b)}{kL\ K(\sqrt{1 - (a/b)^2})} \qquad (64)$$

where L is the length of the system. This coordinate system also is appropriate for the thermal problems shown in Fig. 21. The thermal resistances are, respectively,

$$R = \frac{K(a/b)}{2\,kL\ K(\sqrt{1 - (a/b)^2})} \qquad (65)$$

and, for Fig. 21b, the resistance is twice the value given by Eq. (65). Kutateladze[15] gives the following expression for the resistance of the problem shown in Fig. 21a:

$$R = \frac{0.42}{kL\ [(b/a) - 1]^{0.24}} \qquad (66)$$

Equation (66) is compared with the exact expression [Eq. (65)] in Table 1.

Table 1 Comparison of exact solution and Eq. (66)

a/b	kLR: Eq. (65)	kLR: Eq. (66)	% error
0.1	0.2341	0.2479	5.9
0.2	0.2918	0.30113	3.2
0.3	0.3383	0.34272	1.3
0.4	0.3794	0.3811	0.44
0.5	0.4177	0.4200	0.56
0.6	0.4556	0.4629	1.6
0.7	0.4968	0.5147	3.6
0.8	0.5487	0.5858	6.7
0.9	0.6356	0.7117	12.0

It can be seen that the approximate solution of Kutateladze based upon machine calculation is in error by more than 1% for a/b < 0.3 and for a/b > 0.5. The error is largest for small and large values of a/b. The use of the appropriate coordinate system made the problem amenable to the use of the general expression for the conduction form factor given by Eq. (22).

Conclusion

A general expression for predicting conduction shape factors was developed for cylindrical and rotational systems that are simply separable. The analysis assumed that the thermal problem can always be posed as a one-dimensional thermal problem if the appropriate coordinate system is chosen. The general expression was used to generate several expressions for conduction shape factors based upon some important coordinate systems and to show how the conduction shape factors are related to the geometry of the thermal system. Knowing only the metric or Lamé's coefficients of a system, one can easily determine its conduction shape factor. Several examples of important thermal problems were used to illustrate the use of the conduction shape factors given in the section entitled "General Expressions for Conduction Shape Factors."

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