

# Influence of Geometry and Edge Cooling on Thermal Spreading Resistance

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This paper presents a simple geometric transformation for predicting thermal spreading resistance in isotropic and compound rectangular flux channels using the solution for an isotropic or compound circular flux tube. It is shown that the results are valid for a wide range of channel aspect ratios and source to base coverage ratio. Because the circular disk solution requires a single series summation, it is preferable to the rectangular flux channel solution, which requires the evaluation of two single-series and one double-series summation. The effect of edge cooling is also addressed in flux tubes and flux channels. A new analytical solution is obtained for thermal spreading resistance in a rectangular flux channel with edge cooling. This solution contains many limiting cases, including a previously published solution for adiabatic edges. Comparisons are made with the circular flux tube with edge cooling and with adiabatic edges. Simple relationships are developed for edge-cooled systems to assess the importance of edge cooling. This alleviates the issue of computing or recomputing eigenvalues when the edge-cooling conditions change or have no impact. It is shown that this simple approach provides good results for a wide range of dimensionless parameters.

## Nomenclature

$A_b$	= baseplate area, m <sup>2</sup>
$A_n, B_n$	= Fourier coefficients
$A_{mn}, B_{mn}$	= Fourier coefficients
$A_s$	= heat source area, m <sup>2</sup>
$a, b$	= radial dimensions, m
$a, b, c, d$	= linear dimensions, m
$Bi$	= Biot number, $ht/k$
$Bi_e$	= Biot number, $h_e b/k$
$Bi_{e,x}$	= Biot number, $h_{e,x} c/k$
$Bi_{e,y}$	= Biot number, $h_{e,y} d/k$
$h$	= contact conductance or film coefficient, W/m <sup>2</sup> · K
$J_0(\cdot), J_1(\cdot)$	= Bessel functions of first kind, orders 0 and 1
$k$	= thermal conductivity, W/m · K
$\mathcal{L}$	= length scale $\sqrt{A_s}$ , m
$m, n$	= indices for summations
$Q$	= heat flow rate $q_s A_s$ , W
$q_s$	= heat flux, W/m <sup>2</sup>
$R$	= thermal resistance, K/W
$R_s$	= spreading resistance, K/W
$R_t$	= total resistance, K/W
$R_{1D}$	= one-dimensional resistance, K/W
$R^*$	= dimensionless resistance, $kR\mathcal{L}$
$T$	= temperature, K
$T_f$	= sink temperature, K
$T_s$	= mean source temperature, K

$t, t_1, t_2$	= total and layer thicknesses, m
$\alpha$	= equation parameter $(1 - \kappa)/(1 + \kappa)$
$\beta_{mn}$	= eigenvalues $\sqrt{(\delta_m^2 + \lambda_n^2)}$ and $\sqrt{(\lambda_{xm}^2 + \lambda_{yn}^2)}$
$\delta_m$	= eigenvalues $(m\pi/c)$
$\delta_{xm}, \delta_{yn}, \delta_n$	= eigenvalues
$\epsilon$	= relative source size $a/b$
$\epsilon_b$	= baseplate aspect ratio $c/d$
$\epsilon_x$	= relative source size $a/c$
$\epsilon_y$	= relative source size $b/d$
$\zeta$	= dummy variable $m^{-1}$
$\theta$	= temperature excess $T - T_f$ , K
$\bar{\theta}$	= mean temperature excess $T - T_f$ , K
$\kappa$	= relative conductivity $k_2/k_1$
$\lambda_n$	= eigenvalues $n\pi/d$
$\lambda_{xm}, \lambda_{yn}$	= eigenvalues
$\xi$	= subvariable, Eq. (26)
$\tau$	= relative thickness $t/\mathcal{L}$
$\phi, \varphi$	= spreading resistance functions
$\psi$	= dimensionless spreading parameter $4kaR_s$
$\varrho$	= equation parameter $(\zeta + h/k_2)/(\zeta - h/k_2)$

## Subscripts

$b$	= base
$e$	= equivalent
eff	= effective
$f$	= fluid
$m, n$	= $m$ th and $n$ th terms
$s$	= source
$t$	= total
$x$	= $x$ dir
$y$	= $y$ dir

## Introduction

Thermal spreading resistance in rectangular flux channels is of interest to electronic packaging engineers working with discrete heat sources in heat sink, circuit board, and many other applications where heat enters through a portion of the contacting surface.<sup>1,2</sup> In this paper, geometric equivalences between the circular disk and rectangular flux channel are established, and the effect of edge cooling is examined.

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A review of the literature shows that a number of useful solutions for rectangular flux channels has been obtained for a variety of configurations.<sup>3–8</sup> However, these solutions, which are based on Fourier series expansions,<sup>9,10</sup> require the evaluation of single and double summations. Although these solutions are not computationally intractable in the present day, a solution requiring the evaluation of a single series is much more efficient for practitioners. Such is the case for thermal spreading resistances in circular flux tubes.<sup>11,12</sup>

This paper will demonstrate that a simple geometric equivalence can be used for predicting thermal spreading resistances in rectangular flux channels using the solution for an equivalent circular flux tube. It will be shown that the solution is a weak function of shape and aspect ratio of the heat source and substrate. Theoretical results will be presented for a range of parameters, and simple expressions will be developed to assist in the computations.

Further, a review of the literature also shows that a number of useful solutions for rectangular flux channels has been obtained for a variety of configurations including compound and isotropic flux channels, single and multiple eccentric heat sources, and orthotropic spreaders.<sup>1–8</sup> One issue not yet examined is the effect of edge cooling. This issue was recently addressed for circular flux tubes.<sup>13</sup>

This paper addresses the issue of edge cooling in rectangular flux channels by presenting a new solution. Further, simple expressions are established to show the relative importance of edge cooling in thermal resistance calculations. This is done for both the circular flux tube and rectangular flux channel. The need for a simple predictive approach for edge-cooled systems is motivated by the fact that for each unique value of edge heat-transfer coefficient a unique set of eigenvalues must be tabulated, making computations more tedious. However, this is not the case for systems with adiabatic edges. Theoretical results will be presented for a range of parameters.

### Problem Statement

Thermal spreading resistance arises in multidimensional applications where heat enters a domain through a finite area (refer to Figs. 1 and 2). In typical applications, the system is idealized as having a central heat source placed on one of the heat spreader surfaces, while the lower surface is cooled with a constant conductance, which can represent a heat sink, contact conductance, or convective heat-transfer coefficient. All edges are assumed to be adiabatic or edge cooled. Further, the region outside the heat source in the source plane is also assumed to be adiabatic.

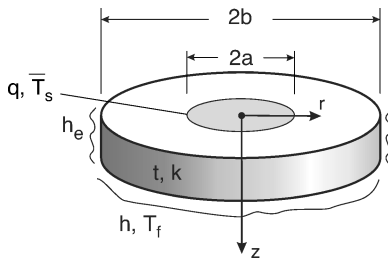


Fig. 1 Circular flux tube with edge cooling.

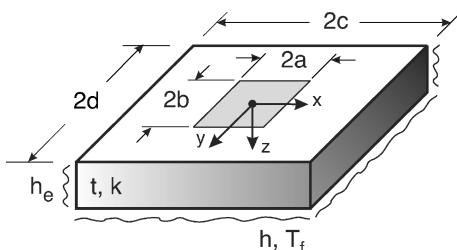


Fig. 2 Rectangular flux channel with edge cooling.

In this idealized system, the total thermal resistance of the system is defined as

$$R_t = (\bar{T}_s - T_f)/Q = \bar{\theta}_s/Q \quad (1)$$

where  $\bar{\theta}_s$  is the mean source temperature excess,  $T_f$  is the sink temperature, and  $Q$  is the total heat input of the device over the contact region. The mean source temperature is given by

$$\bar{\theta}_s = \frac{1}{A_s} \iint_{A_s} \theta(x, y, 0) dA_s \quad (2)$$

In applications involving adiabatic edges, the total thermal resistance is composed of two terms: a uniform flow or one-dimensional resistance and a spreading or multidimensional resistance, which vanishes as the source area approaches the substrate area, that is,  $A_s \rightarrow A_b$ . These two components are combined as follows:

$$R_t = R_{1D} + R_s \quad (3)$$

Thermal spreading resistance analysis of a rectangular spreader requires the solution of Laplace's equation

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \quad (4)$$

in three dimensions, and for circular disk spreaders

$$\nabla^2 T = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \quad (5)$$

in two dimensions.

In most applications the following boundary conditions are applied:

$$\left. \begin{aligned} \frac{\partial T}{\partial n} \Big|_{x=0, y=0, r=0} &= 0, & n = x, y, r \\ \frac{\partial T}{\partial n} \Big|_{x=c, y=d, r=b} &= 0, & n = x, y, r \end{aligned} \right\} \quad (6)$$

along the edges,  $x=c$ ,  $y=d$ , or  $r=b$ , and at the centroid of the substrate,  $x=0$ ,  $y=0$ , or  $r=0$ . Over the top surface  $z=0$ ,

$$\left. \begin{aligned} \frac{\partial T}{\partial z} \Big|_{z=0} &= 0, & A_s < A < A_b \\ \frac{\partial T}{\partial z} \Big|_{z=0} &= -\frac{q_s}{k}, & 0 < A < A_s \end{aligned} \right\} \quad (7)$$

where  $A_s$  is the area of the heat source and  $A_b$  is the area of the base or substrate. Finally, along the lower surface  $z=t$ ,

$$\frac{\partial T}{\partial z} \Big|_{z=t} + \frac{h}{k} [T(x, y, t) - T_f] = 0 \quad (8)$$

where  $h$  is a uniform convection heat-transfer coefficient or contact conductance.

In many systems edge cooling can be a significant factor. In the present solution all edges are assumed to be cooled with a constant edge heat-transfer coefficient  $h_e$  in the case of a circular flux tube, or different edge coefficients  $h_{e,x}$  and  $h_{e,y}$  in the case of a rectangular flux channel (refer to Figs. 1 and 2):

$$r = b, \quad \frac{\partial \theta}{\partial r} + \frac{h_e}{k} \theta = 0 \quad (9)$$

or

$$\left. \begin{aligned} x = c, & \quad \frac{\partial \theta}{\partial x} + \frac{h_{e,x}}{k} \theta = 0 \\ y = d, & \quad \frac{\partial \theta}{\partial y} + \frac{h_{e,y}}{k} \theta = 0 \end{aligned} \right\} \quad (10)$$

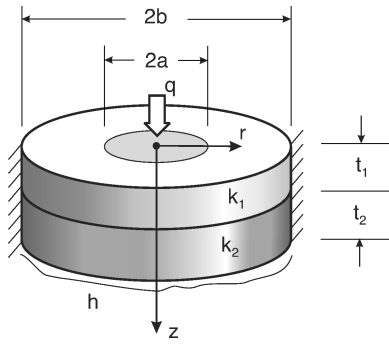


Fig. 3 Compound disk with circular heat source.

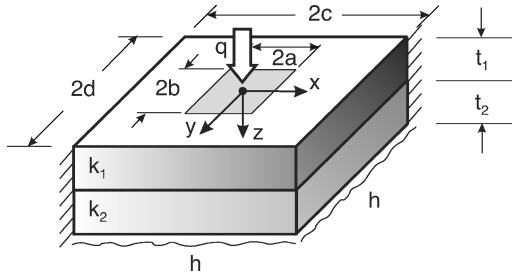


Fig. 4 Compound flux channel with rectangular heat source.

where  $h_{e,x}$  and  $h_{e,y}$  denote the values of the edge heat-transfer coefficient, along the  $x$  edge and  $y$  edge, respectively.

When edge cooling is present, the resistance remains multidimensional for all conditions except for the special case of  $h_e = 0$ .

In compound systems (refer to Figs. 3 and 4), Laplace's equation must be written for each layer in the system, and continuity of temperature and heat flux at the interface is required, yielding two additional boundary conditions:

$$\left. \begin{aligned} T_1(x, y, t_1) &= T_2(x, y, t_1) \\ k_1 \frac{\partial T_1}{\partial z} \Big|_{z=t_1} &= k_2 \frac{\partial T_2}{\partial z} \Big|_{z=t_1} \end{aligned} \right\} \quad (11)$$

For a compound system Eq. (8) is written to represent  $T_2$ , that is,

$$\frac{\partial T_2}{\partial z} + \frac{h}{k}(T_2 - T_f) = 0 \quad (12)$$

### Geometric Equivalence

The necessary solutions for the systems to be examined are found in the papers by Yovanovich et al.<sup>3,11</sup> They are given in the following for the sake of completeness because they will be nondimensionalized in a more appropriate manner. Thermal spreading resistance solutions in isotropic and compound disks (refer to Figs. 3–6), flux channels, and half-spaces are presented in Yovanovich et al.<sup>11,12</sup> A general solution for the compound disk was first obtained by Yovanovich et al.<sup>12</sup> The general solution<sup>12</sup> is

$$4k_1 a R_s = \frac{8}{\pi \epsilon} \sum_{n=1}^{\infty} A_n(n, \epsilon) B_n(n, \tau, \tau_1) \frac{J_1(\delta_n \epsilon)}{\delta_n \epsilon} \quad (13)$$

where

$$A_n = -\frac{2\epsilon J_1(\delta_n \epsilon)}{\delta_n^2 J_0^2(\delta_n)} \quad (14)$$

and

$$B_n = \frac{\phi_n \tanh(\delta_n \tau_1) - \varphi_n}{1 - \phi_n} \quad (15)$$

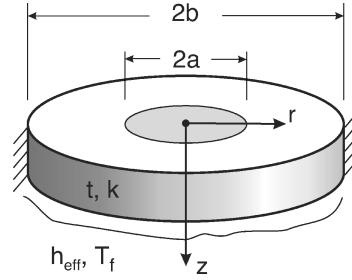


Fig. 5 Isotropic disk with circular heat source.

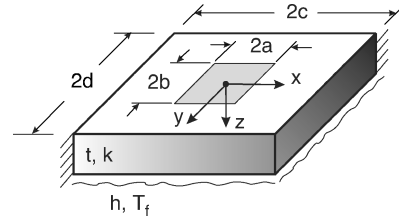


Fig. 6 Isotropic flux channel with rectangular heat source.

The functions  $\phi_n$  and  $\varphi_n$  are defined as follows:

$$\phi_n = \frac{\kappa - 1}{\kappa} \cosh(\delta_n \tau_1) [\cosh(\delta_n \tau_1) - \varphi_n \sinh(\delta_n \tau_1)] \quad (16)$$

and

$$\varphi_n = \frac{\delta_n + Bi \tanh(\delta_n \tau)}{\delta_n \tanh(\delta_n \tau) + Bi} \quad (17)$$

The eigenvalues  $\delta_n$  are roots of  $J_1(\delta_n) = 0$  and  $Bi = hb/k_2$ ,  $\tau = t/b$ , and  $\tau_1 = t_1/b$ .

Thermal spreading resistance in rectangular systems was recently obtained by the authors.<sup>3</sup> In Yovanovich et al.,<sup>3</sup> the authors obtained a general solution for a compound rectangular flux channel having a central heat source (refer to Fig. 4). This general solution also simplifies for many cases of semi-infinite flux channels and half-space solutions.<sup>3</sup> More recently, the authors<sup>4</sup> developed a solution for a single eccentric heat source on compound and isotropic flux channels. The results of Muzychka et al.<sup>4</sup> were also extended to systems having multiple arbitrarily located heat sources.

The spreading resistance of Yovanovich et al.<sup>3</sup> is obtained from the following general expression according to the notation in Fig. 4:

$$\begin{aligned} R_s &= \frac{c^2}{2k_1 a^2 d} \sum_{m=1}^{\infty} \frac{\sin^2(m\pi a/c)}{(m\pi)^3} \cdot \varphi(\delta_m) \\ &+ \frac{d^2}{2k_1 b^2 c} \sum_{n=1}^{\infty} \frac{\sin^2(n\pi b/d)}{(n\pi)^3} \cdot \varphi(\lambda_n) \\ &+ \frac{cd}{k_1 a^2 b^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin^2(m\pi a/c) \sin^2(n\pi b/d)}{(m\pi)^2 (n\pi)^2 \beta_{mn}} \cdot \varphi(\beta_{mn}) \end{aligned} \quad (18)$$

where

$$\varphi(\zeta) = \frac{(\alpha e^{4\zeta t_1} + e^{2\zeta t_1}) + \varrho [e^{2\zeta(2t_1+t_2)} + \alpha e^{2\zeta(t_1+t_2)}]}{(\alpha e^{4\zeta t_1} - e^{2\zeta t_1}) + \varrho [e^{2\zeta(2t_1+t_2)} - \alpha e^{2\zeta(t_1+t_2)}]} \quad (19)$$

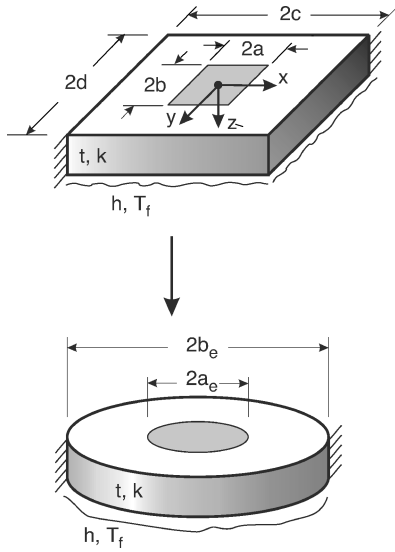
and

$$\varrho = \frac{\zeta t + ht/k_2}{\zeta t - ht/k_2} \quad \text{and} \quad \alpha = \frac{1 - \kappa}{1 + \kappa}$$

with  $\kappa = k_2/k_1$ . The eigenvalues for these solutions are  $\delta_m = m\pi/c$ ,  $\lambda_n = n\pi/d$ , and  $\beta_{mn} = \sqrt{\delta_m^2 + \lambda_n^2}$  and are denoted by  $\zeta$  in Eq. (19). Equation (19) simplifies for an isotropic disk to give

$$\varphi_n = \frac{\zeta t + ht/k_1 \tanh(\zeta t)}{\zeta t \tanh(\zeta t) + ht/k_1} \quad (20)$$

where  $k_1$  is now the thermal conductivity of the flux channel.



**Fig. 7** Equivalence between a rectangular flux channel and circular flux tube.

Next, these two solutions are compared, and it is shown that considerable computational effort is saved by modeling the rectangular flux channel as an equivalent circular flux tube for a wide range of channel aspect ratios.

Given the two solutions for the circular disk and rectangular flux channel, it is now possible to show that there exists a geometric equivalence between the two systems as shown in Fig. 7. Geometric equivalence was established by Muzychka et al.<sup>14</sup> for computing the spreading resistance in annular sectors. In determining this equivalence, we choose to maintain the volume of material in the system. In doing so, the following conditions must be satisfied:

$$A_{s,R} = A_{s,C} \quad A_{b,R} = A_{b,C} \quad t_R = t_C \quad (21)$$

or

$$a_e = \sqrt{A_s/\pi} \quad b_e = \sqrt{A_b/\pi} \quad t = t \quad (22)$$

This is equivalent to nondimensionalizing the circular disk solution using the characteristic length scale  $\mathcal{L} = \sqrt{A_s} = \sqrt{\pi}a$ . This length scale has been utilized in the past.<sup>15–17</sup>

This leads to the following dimensionless spreading resistance defined as  $R_s^* = k\sqrt{A_s}R_s$  for an isotropic circular flux tube<sup>11,17</sup>:

$$R_s^* = \frac{4}{\sqrt{\pi}\epsilon} \sum_{n=1}^{\infty} \frac{J_1^2(\delta_n\epsilon)}{\delta_n^3 J_0^2(\delta_n)} \cdot \phi_n \quad (23)$$

where

$$\phi_n = \frac{\delta_n\epsilon\sqrt{\pi}\tau + Bi \tanh(\delta_n\epsilon\sqrt{\pi}\tau)}{Bi + \delta_n\epsilon\sqrt{\pi}\tau \tanh(\delta_n\epsilon\sqrt{\pi}\tau)} \quad (24)$$

The one-dimensional resistance can be written in dimensionless form as

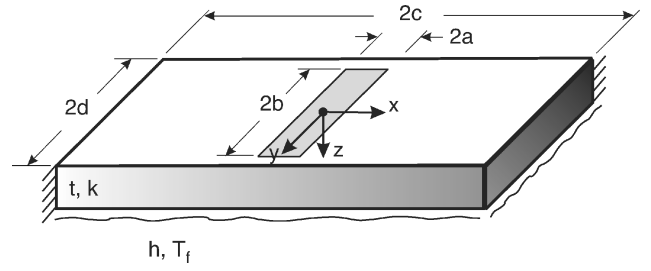
$$R_{1D}^* = (1 + 1/Bi)\tau\epsilon^2 \quad (25)$$

Thus, the spreading, one-dimensional, and total resistances are a function of

$$R^* = f(\epsilon, \tau, Bi) \quad (26)$$

where  $\tau = t/\sqrt{A_s}$ ,  $Bi = ht/k$ , and  $\epsilon = \sqrt{(A_s/A_b)} = a/b$ .

The solution for spreading resistance in a rectangular flux channel can be nondimensionalized in a similar manner using



**Fig. 8** Nonconforming source/substrate system.

$\mathcal{L} = \sqrt{A_s} = 2\sqrt{(ab)}$ . The resulting expression is written as follows:

$$R_s^* = \sum_{m=1}^{\infty} \frac{\sin^2(m\pi\epsilon_x)\sqrt{\epsilon_x\epsilon_y\epsilon_b}}{(m\pi)^3\epsilon_x^2} \cdot \phi_x \\ + \sum_{n=1}^{\infty} \frac{\sin^2(n\pi\epsilon_y)\sqrt{\epsilon_x\epsilon_y\epsilon_b}}{(n\pi)^3\epsilon_y^2\epsilon_b} \cdot \phi_y \\ + 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin^2(m\pi\epsilon_x)\sin^2(n\pi\epsilon_y)\sqrt{\epsilon_x\epsilon_y\epsilon_b}}{\epsilon_x^2\epsilon_y^2m^2n^2\pi^5\sqrt{m^2/\epsilon_b^2 + n^2}} \cdot \phi_{xy} \quad (27)$$

This expression is valid for the system depicted in Fig. 8. If the flux channel is rotated 90 deg, that is, interchange  $x$  and  $y$  values, it gives the same value for the spreading resistance. The spreading functions  $\phi$ , which account for the effects of the conductance  $h$  and finite thickness  $t$ , are

$$\phi(\xi) = \frac{\xi\tau + Bi \tanh(\xi\tau)}{Bi + \xi\tau \tanh(\xi\tau)} \quad (28)$$

where

$$\left. \begin{aligned} \phi_x &= \phi(\xi \rightarrow 2\pi m\sqrt{\epsilon_x\epsilon_y\epsilon_b}/\epsilon_b) \\ \phi_y &= \phi(\xi \rightarrow 2\pi n\sqrt{\epsilon_x\epsilon_y\epsilon_b}) \\ \phi_{xy} &= \phi(\xi \rightarrow 2\pi\sqrt{m^2/\epsilon_b^2 + n^2}\sqrt{\epsilon_x\epsilon_y\epsilon_b}) \end{aligned} \right\} \quad (29)$$

The one-dimensional resistance can be written in dimensionless form as

$$R_{1D}^* = (1 + 1/Bi)\tau\epsilon_x\epsilon_y \quad (30)$$

Thus, the dimensionless spreading, one-dimensional, and total resistances are now a function of

$$R_s^* = f(\epsilon_x, \epsilon_y, \epsilon_b, \tau, Bi) \quad (31)$$

where  $\epsilon_x = a/c$ ,  $\epsilon_y = b/d$ ,  $\epsilon_b = c/d$ ,  $\tau = t/\sqrt{A_s}$ , and  $Bi = ht/k$ .

Equation (27) simplifies for the special case when  $\epsilon_b = 1$ , that is, a square flux channel. Similar results can be obtained for compound disks and flux channels. Comparisons will be made for isotropic systems, but the analysis is also valid for compound systems.

We can now compare the solution for the disk and the flux channel using Eqs. (23) and (24) and (27) and (28). Two configurations are examined. These are the square flux channel with a central square heat source and a rectangular flux channel with a rectangular heat source, which might or might not conform to the aspect ratio of the flux channel as shown in Fig. 8. In the case of the flux tube solution, 200 terms were used in the summation in Eq. (23). In the case of the flux channel, 200 terms were used in each of the single summations and 50 by 50 terms in the double summation of Eq. (27). This provides accuracy to at least four decimal places.

First, for the special case of a square flux channel with a square heat source, the solution simplifies considerably because  $\epsilon_b = 1$  and  $\epsilon_x = \epsilon_y = \epsilon$ . The solutions for both the circular disk and square flux channel are given in Figs. 9–11 for a range of dimensionless thickness  $\tau$ ,  $Bi$ , and  $\epsilon$ . Excellent agreement is obtained for these two

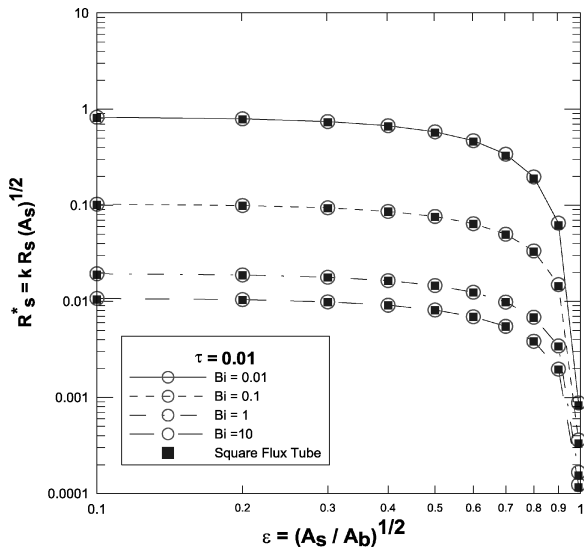


Fig. 9 Equivalence for a square channel for  $\tau = 0.001$ .

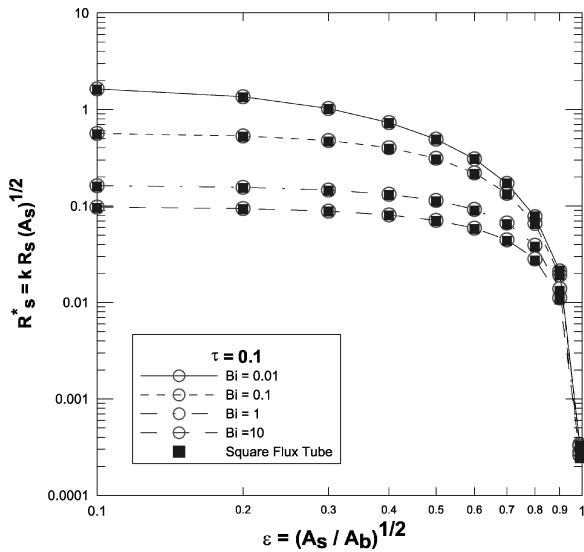


Fig. 10 Equivalence for a square channel for  $\tau = 0.1$ .

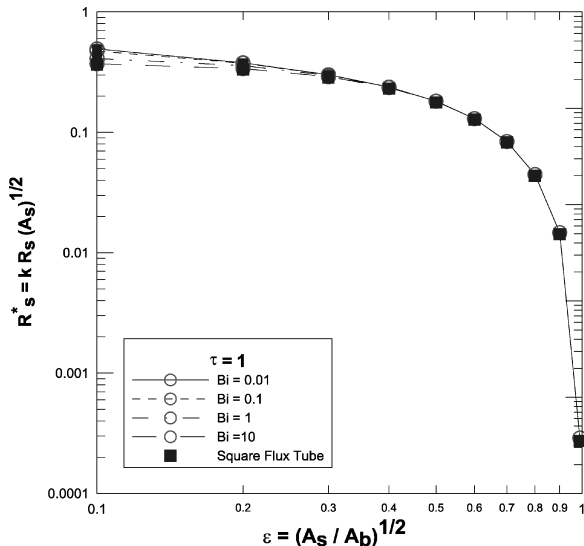


Fig. 11 Equivalence for a square channel for  $\tau = 1$ .

cases, as can be seen in Figs. 9–11. It is clear that preserving the volume of the material, by transforming the square flux channel into a circular flux tube, gives equivalent results for the dimensionless spreading resistance at the same coverage ratio  $\epsilon = \sqrt{A_s/A_b}$ .

Next we examine the effect of flux channel and heat source aspect ratios. Three flux channel aspect ratios are examined:  $\epsilon_b = c/d = 1, 2, 4$ . Equivalent results are obtained if  $\epsilon_b = c/d = 1, \frac{1}{2}, \frac{1}{4}$ . For each of these cases, the source aspect ratio is varied such that  $\epsilon_x = 0.2, 0.4, 0.6, 0.8$  and  $\epsilon_y = 0.2, 0.4, 0.6, 0.8$ . This leads to 16 combinations for each flux channel considered.<sup>18</sup> Further, we have also considered three dimensionless thicknesses  $\tau = 0.01, 0.1, 1$  and two Biot numbers  $Bi = 10, 100$ . To compare the circular flux tube and the rectangular flux channel, the results must be plotted using the common aspect ratio defined as

$$\epsilon = \sqrt{A_s/A_b} = \sqrt{\epsilon_x \epsilon_y} \tag{32}$$

Thus for each case examined an equivalent circular aspect ratio is determined from  $\epsilon_x$  and  $\epsilon_y$ . This leads to nine unique predictions using the flux tube solution. Therefore, for some aspect ratios two solutions are determined for the flux channel as a result of source orientation, while there is only one equivalent flux tube solution. The results of this comparison are presented in Figs. 12–14. It is

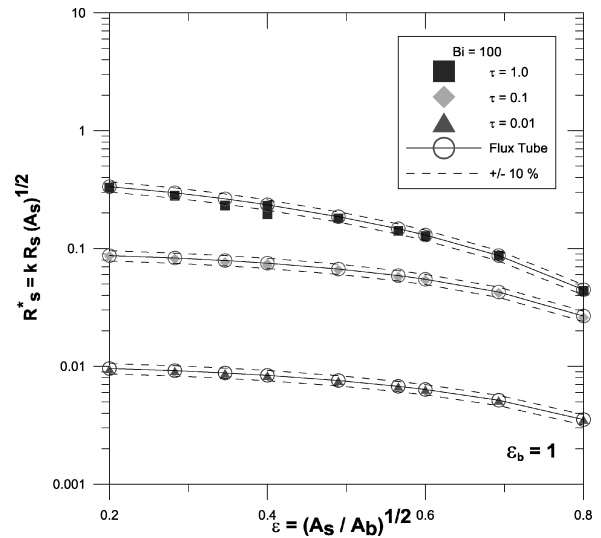


Fig. 12 Equivalence model for  $Bi = 100$  and  $\epsilon_b = 1$ .

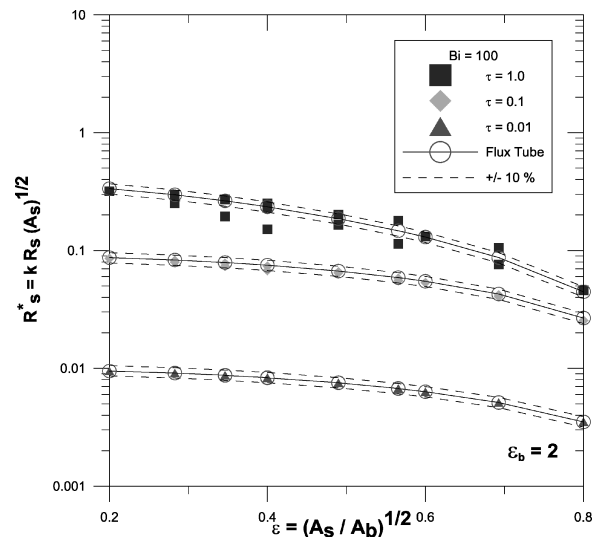


Fig. 13 Equivalence model for  $Bi = 100$  and  $\epsilon_b = 2$ .

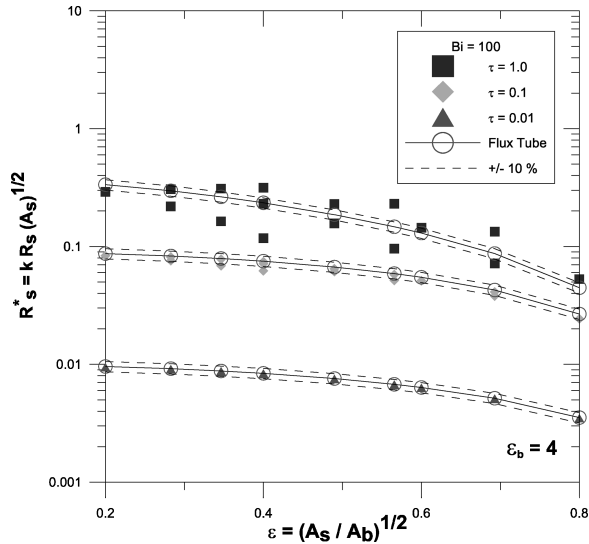


Fig. 14 Equivalence model for  $Bi = 100$  and  $\epsilon_b = 4$ .

clear that for most combinations with  $\tau < 1$  there exists an equivalence between the two systems for which the error is much less than  $\pm 10\%$ . The accuracy increases as  $Bi$  increases and as  $\tau$  decreases for all values of  $\epsilon$  considered. In general, the accuracy for conforming rectangular systems exceeds that of similar nonconforming systems.

The error in the total resistance will be much less once the one-dimensional resistance is combined with the spreading resistance. Thus the present approach allows for a simple and convenient method for computing the thermal resistance in rectangular flux channels using the circular flux tube solution. Additional results are presented graphically in Muzychka et al.<sup>18</sup>

### Systems with Edge Cooling

The solution for the total thermal resistance  $R_t$  for the general system shown in Fig. 1 was recently obtained by one of the authors.<sup>13</sup> It can be written in the following dimensionless form:

$$R_t^* = \frac{4}{\sqrt{\pi}\epsilon} \sum_{n=1}^{\infty} \frac{J_1^2(\delta_n \epsilon) \phi_n}{\delta_n^3 [J_0^2(\delta_n) + J_1^2(\delta_n)]} \quad (33)$$

and

$$\phi_n = \frac{\delta_n \epsilon \sqrt{\pi} \tau + Bi \tanh(\delta_n \epsilon \sqrt{\pi} \tau)}{Bi + \delta_n \epsilon \sqrt{\pi} \tau \tanh(\delta_n \epsilon \sqrt{\pi} \tau)} \quad (34)$$

where  $R_t^* = R_t k \sqrt{A_s}$ ,  $\tau = t / \sqrt{A_s}$ ,  $Bi = ht / k$ ,  $\epsilon = \sqrt{(A_s / A_b)} = a / b$ , and  $\delta_n$  are the eigenvalues. The eigenvalues are obtained from application of the second boundary condition along the disk edges and require numerical solution to the following transcendental equation:

$$\delta_n J_1(\delta_n) = Bi_e J_0(\delta_n) \quad (35)$$

where  $\delta_n = \lambda_n b$ ,  $Bi_e = h_e b / k$  is the edge Biot number, and  $J_0(\cdot)$  and  $J_1(\cdot)$  are Bessel functions of the first kind of order zero and one, respectively. A unique set of eigenvalues must be computed for each value of  $Bi_e$ . Simplified expressions for predicting the eigenvalues were developed by Yovanovich<sup>13</sup> using the Newton–Raphson method.

It is now clear that the dimensionless total resistance depends upon

$$R_t^* = f(\epsilon, \tau, Bi_e, Bi) \quad (36)$$

whereas the dimensional total resistance depends upon

$$R_t = f(a, b, t, h_e, h) \quad (37)$$

In the case of a rectangular flux channel, no solution exists for edge cooling. Thermal spreading resistance in a rectangular flux channel with adiabatic edges was recently obtained by the authors.<sup>3</sup> The solution methodology is the same, and the resulting solution is quite similar, with the exception of the definition of the eigenvalues. The solution for an isotropic flux channel with edge cooling can be obtained by means of separation of variables.<sup>19,20</sup> The solution is assumed to have the form  $\theta(x, y, z) = X(x) * Y(y) * Z(z)$ , where  $\theta(x, y, z) = T(x, y, z) - T_f$ . Applying the method of separation of variables yields the following general solution for the temperature excess in the substrate, which satisfies the thermal boundary conditions along the two planes of symmetry,  $x = 0$  and  $y = 0$ :

$$\theta(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \cos(\lambda_{xm} x) \cos(\lambda_{yn} y) \times [A_{mn} \cosh(\beta_{mn} z) + B_{mn} \sinh(\beta_{mn} z)] \quad (38)$$

where  $\lambda_{xm}$ ,  $\lambda_{yn}$ , and  $\beta_{mn} = \sqrt{(\lambda_{xm}^2 + \lambda_{yn}^2)}$  are the eigenvalues. The eigenvalues are obtained from the following equations:

$$\delta_{xm} \sin(\delta_{xm}) = Bi_{e,x} \cos(\delta_{xm}) \quad (39)$$

and

$$\delta_{yn} \sin(\delta_{yn}) = Bi_{e,y} \cos(\delta_{yn}) \quad (40)$$

where  $Bi_{e,x} = h_{e,x} c / k$ ,  $\delta_{xm} = \lambda_{xm} c$ ,  $Bi_{e,y} = h_{e,y} d / k$ , and  $\delta_{yn} = \lambda_{yn} d$ . These equations must be solved numerically for a finite number of eigenvalues for each specified value of the edge cooling Biot numbers. The separation constant  $\beta_{mn}$  is now defined as

$$\beta_{mn} = \sqrt{(\delta_{xm} / c)^2 + (\delta_{yn} / d)^2} \quad (41)$$

Application of the lower surface boundary condition yields the following relation:

$$A_{mn} = -B_{mn} \cdot \phi_{mn} \quad (42)$$

where

$$\phi_{mn} = \frac{t \beta_{mn} + (ht / k) \tanh(\beta_{mn} t)}{ht / k + t \beta_{mn} \tanh(\beta_{mn} t)} \quad (43)$$

The final Fourier coefficients are obtained by taking a double Fourier expansion of the upper surface condition. This yields the following expression:

$$B_{mn} = \frac{-Q \int_0^a \cos(\lambda_{xm} x) dx \int_0^b \cos(\lambda_{yn} y) dy}{4kab \beta_{mn} \int_0^c \cos^2(\lambda_{xm} x) dx \int_0^d \cos^2(\lambda_{yn} y) dy} \quad (44)$$

Upon evaluation of the integrals, one obtains

$$B_{mn} = \frac{-Q \sin(\delta_{xm} a / c) \sin(\delta_{yn} b / d)}{kab \beta_{mn} [\sin(2\delta_{xm}) / 2 + \delta_{xm}] [\sin(2\delta_{yn}) / 2 + \delta_{yn}]} \quad (45)$$

With both Fourier coefficients now known, the mean surface temperature excess is found from Eq. (2). Using this result and Eq. (1), the total resistance becomes

$$R_t = \frac{cd}{ka^2 b^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin^2(\delta_{xm} a / c) \sin^2(\delta_{yn} b / d) \phi_{mn}}{\delta_{xm} \delta_{yn} \beta_{mn} [\sin(2\delta_{xm}) / 2 + \delta_{xm}] [\sin(2\delta_{yn}) / 2 + \delta_{yn}]} \quad (46)$$

The total resistance now depends on

$$R_t = f(a, b, c, d, t, k, h, h_{e,x}, h_{e,y}) \quad (47)$$

The total resistance is nondimensionalized using  $\mathcal{L} = \sqrt{A_s} = 2\sqrt{(ab)}$  to give

$$R_t^* = \frac{2\sqrt{\epsilon_x \epsilon_y \epsilon_b}}{\epsilon_b \epsilon_x^2 \epsilon_y^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\delta_{xm} \delta_{yn} \sqrt{\delta_{xm}^2 / \epsilon_b^2 + \delta_{yn}^2}} \times \frac{\sin^2(\delta_{xm} \epsilon_x) \sin^2(\delta_{yn} \epsilon_y) \phi_{mn}}{[\sin(2\delta_{xm})/2 + \delta_{xm}][\sin(2\delta_{yn})/2 + \delta_{yn}]} \quad (48)$$

where

$$\phi_{mn} = \frac{\xi \tau + Bi \tanh(\xi \tau)}{Bi + \xi \tau \tanh(\xi \tau)} \quad (49)$$

and

$$\xi = 2\sqrt{\epsilon_x \epsilon_y \epsilon_b} \sqrt{\delta_{xm}^2 / \epsilon_b^2 + \delta_{yn}^2} \quad (50)$$

Thus, the dimensionless total resistance depends upon

$$R_t^* = f(\epsilon_x, \epsilon_y, \epsilon_b, \tau, Bi, Bi_{e,x}, Bi_{e,y}) \quad (51)$$

where

$$\epsilon_x = a/c \quad \epsilon_y = b/d \quad \epsilon_b = c/d$$

$$Bi = ht/k \quad Bi_{e,x} = h_{e,x}c/k \quad Bi_{e,y} = h_{e,y}d/k$$

$$\tau = t/\sqrt{A_s} \quad R_t^* = k\sqrt{A_s}R_t$$

This general result has many geometric special cases. These include semi-infinite flux tubes  $t \rightarrow \infty$ , infinite plate  $c, d \rightarrow \infty$ , half-space<sup>3</sup>  $t, c, d \rightarrow \infty$ , two-dimensional strips  $b=d$  or  $a=c$ , and adiabatic edges<sup>3</sup>  $h_{e,x} \rightarrow 0$  and  $h_{e,y} \rightarrow 0$ . A particularly interesting property is the case when adiabatic edges are present. It can be shown in this case that when  $Bi_{e,x} \rightarrow 0$  and  $Bi_{e,y} \rightarrow 0$  the double summation, which represents the total resistance, consists of the one-dimensional resistance when  $m=n=1$  and the spreading resistance when the remaining terms are summed.

Next, solutions for the edge-cooled flux tube and flux channel are examined, and it is shown that considerable computational effort can be saved by modeling the rectangular system as an equivalent circular disk for a wide range of channel aspect ratios.

Given the solutions for the circular flux tube and rectangular flux channel with edge cooling, it is now possible to show that there exists a physical equivalence between these systems and systems with adiabatic edges (refer to Fig. 15). In determining this equivalence, we choose to maintain the total convective heat-transfer rate at the edges and bottom surface. This is achieved by means of the following energy balance:

$$Q_{\text{total}} = Q_{\text{base}} + Q_{\text{edges}} \quad (52)$$

or

$$h_{\text{eff}} A_b (\bar{T}_b - T_f) = h_b A_b (\bar{T}_b - T_f) + h_e A_e (\bar{T}_e - T_f) \quad (53)$$

For small aspect ratios  $\epsilon$  and thin substrates  $\tau, \bar{T}_e \sim \bar{T}_b$ , which leads to the following condition:

$$h_{\text{eff}} A_b = h A_b + h_e A_e \quad (54)$$

where  $h_{\text{eff}}$  is an effective bottom surface heat-transfer coefficient.

Using the edge and lower surface areas of the disk, we can write

$$h_{\text{eff}} = h + h_e (A_e/A_b) = h + h_e (2t/b) \quad (55)$$

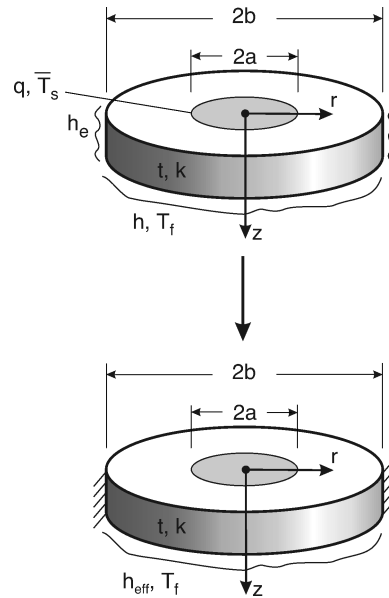


Fig. 15 Circular flux tube with edge cooling transferred to bottom surface.

It can now be seen from this expression that when the edge-cooling coefficient is small and/or the relative thickness and/or relative contact are small the effect of edge cooling is negligible. Similarly, for the rectangular flux channel, we can obtain a similar result, which now contains the two edge-cooling coefficients such that

$$h_{\text{eff}} = h + h_{e,x} (A_{e,x}/A_b) + h_{e,y} (A_{e,y}/A_b) \quad (56)$$

or

$$h_{\text{eff}} = h + h_{e,x} (t/d) + h_{e,y} (t/c) \quad (57)$$

Once again, similar behavior is seen in terms of the relative significance of edge cooling. Finally, it can also be shown that the equivalent rectangular flux channel with edge cooling can be modeled as an equivalent circular disk with edge cooling (refer to Fig. 16) using the earlier results of this paper provided that

$$h_e = \frac{h_{e,x}c + h_{e,y}d}{c + d} \quad (58)$$

in addition to

$$a_e = \sqrt{A_s/\pi} \quad b_e = \sqrt{A_b/\pi} \quad t = t \quad (59)$$

We now examine the influence of edge cooling in the flux tube and/or channel. For simplicity, only the flux tube solution is considered because it only depends upon four variables, whereas the rectangular flux channel depends upon seven variables. In the computations 200 terms were used in Eq. (33) to provide four decimal place accuracy. The solution for the circular flux tube with edge cooling is plotted in Figs. 17–19, for a range of dimensionless thicknesses and source aspect ratio. Nine combinations of lower surface  $Bi$  number and edge  $Bi_e$  number were considered.<sup>21</sup> These are obtained by choosing  $Bi = 1, 10, 100$  and  $Bi_e = 1, 10, 100$ . It is clear from the figures that the total resistance for adiabatic and edge-cooled systems is equivalent, when source aspect ratio is small and relative thickness is small. However, for larger values of edge Biot number and in systems with large source aspect ratios and/or large relative thickness, the effect of edge cooling becomes important and cannot be neglected. Additional results can be found in Muzychka et al.<sup>21</sup>

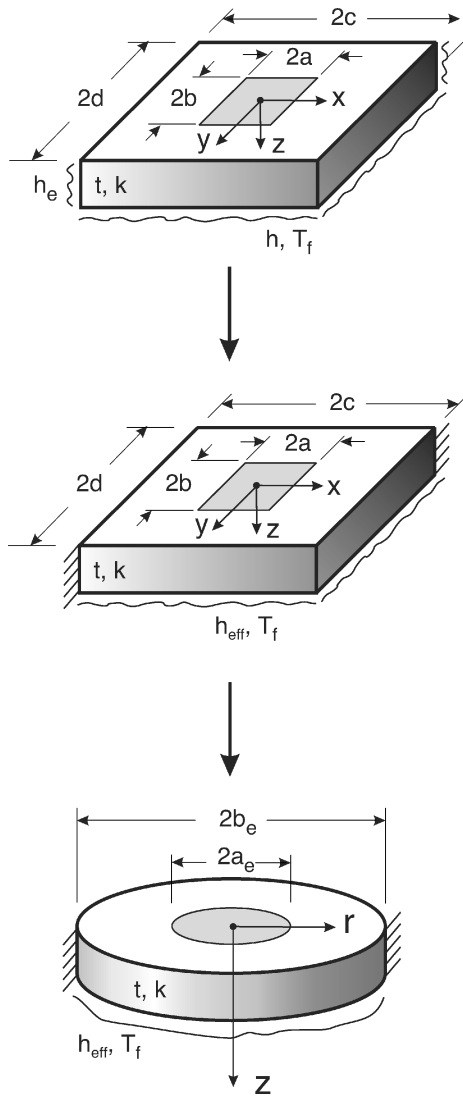


Fig. 16 Rectangular flux channel with edge cooling transferred to bottom surface.

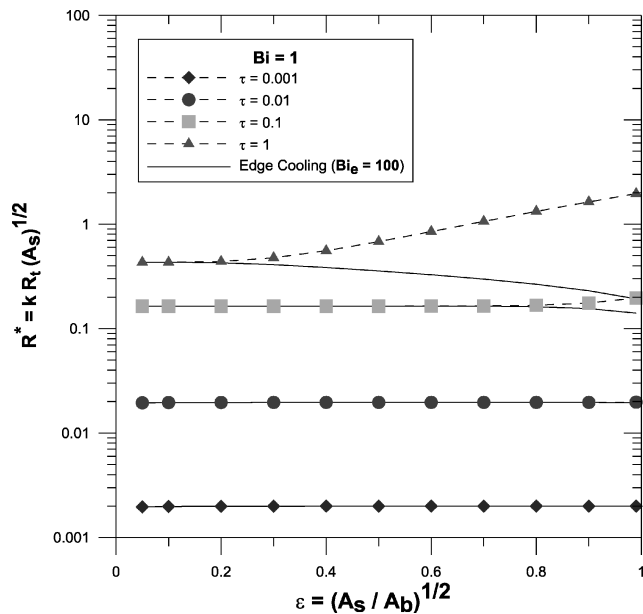


Fig. 17 Comparison of edge-cooled disk for  $Bi = 1, Bi_e = 100$ .

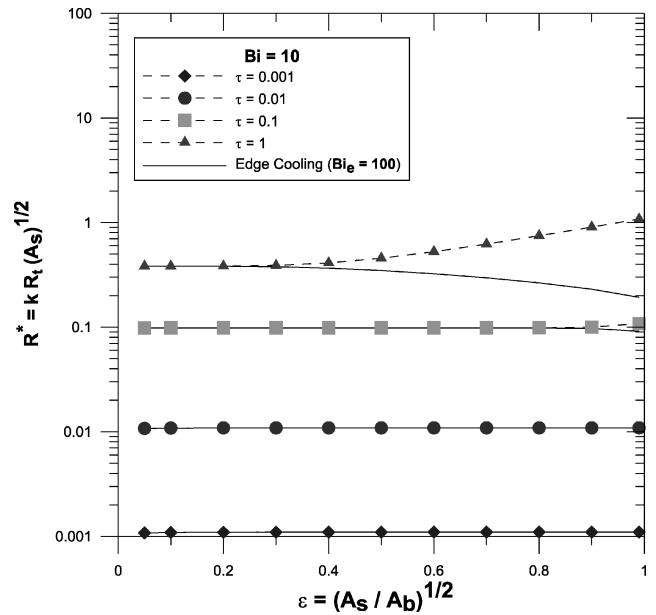


Fig. 18 Comparison of edge-cooled disk for  $Bi = 10, Bi_e = 100$ .

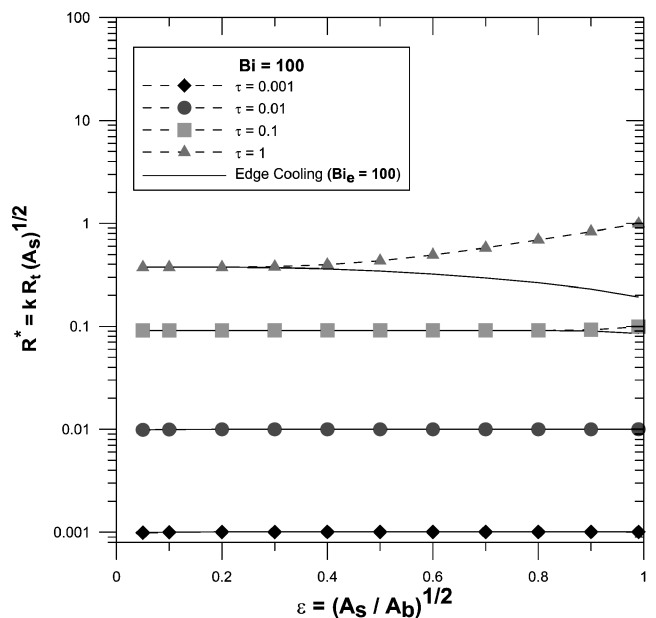


Fig. 19 Comparison of edge-cooled disk for  $Bi = 100, Bi_e = 100$ .

Summary

This paper examined the exact solutions of the circular flux tube and rectangular flux channel for both isotropic and compound systems. It was shown that the solution for the circular flux tube can be used to model the rectangular flux channel when an appropriate geometric equivalence is established. Graphical results were presented for a wide range of system parameters. It was shown that the equivalence is accurate for moderately sized contacts of any aspect ratio. The issue of edge cooling in circular flux tubes and rectangular flux channels was also addressed. A new solution was obtained for the rectangular flux channel with edge cooling. This solution was shown to have many special limiting cases. Simple expressions were developed for predicting the impact of edge cooling. Comparisons were made with the exact solution for edge cooling and the solution for flux tubes with adiabatic edges. This simplified approach allows for efficient computation as the eigenvalues are only computed once in systems with adiabatic edges.



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