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Thermal Spreading Resistance in Compound and Orthotropic Systems

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A review of thermal spreading resistances in compound and orthotropic systems is presented. Solutions for thermal spreading resistances in compound systems are reported. Solutions are reported for both cylindrical and rectangular systems, variable flux distributions, and edge cooling. Transformations of the governing equations and boundary conditions for orthotropic systems are discussed, and new solutions are obtained for rectangular flux channels and circular flux tubes.

Nomenclature

A_b	=	baseplate area, m^2
A_m, A_n, A_{mn}, B_n	=	Fourier coefficients
A_s	=	heat source area, m^2
a, b	=	radial dimensions, m
a, b, c, d	=	linear dimensions, m
Bi	=	Biot number, $h\mathcal{L}/k$
Bi_e	=	Biot number, $h_e b/k$
$Bi_{e,x}$	=	Biot number, $h_{e,x} c/k$
$Bi_{e,y}$	=	Biot number, $h_{e,y} d/k$
h	=	contact conductance or film coefficient, W/m$^2 \cdot$ K
$J_n(\cdot)$	=	Bessel function of order n
$J_0(\cdot), J_1(\cdot)$	=	Bessel function of first kind of order zero and one
k	=	thermal conductivity, W/m $\cdot$ K
k_{eff}	=	effective conductivity, W/m $\cdot$ K
L	=	length of annular sector, m
$\mathcal{L}$	=	arbitrary length scale, m
m, n	=	indices for summations
N	=	number of layers
n	=	outward directed normal
Q	=	heat flow rate, qA_s, W
q	=	heat flux, W/m^2
R	=	thermal resistance, K/W
R_s	=	spreading resistance, K/W
R_T	=	total resistance, K/W
R_{1D}	=	one-dimensional resistance, K/W
R^*	=	dimensionless thermal resistance, kRL
T	=	temperature, K
T_f	=	sink temperature, K
$\bar{T}_s$	=	mean source temperature, K
t, t_1, t_2	=	total and layer thicknesses, m
t_{eff}	=	effective thickness, m
X_c, Y_c	=	heat source centroid, m
α	=	equation parameter, $(1-\kappa)/(1+\kappa)$
α, β	=	angular measurement, rad
β_{mn}	=	eigenvalues, $\sqrt{(\lambda_m^2 + \delta_n^2)}$
$\Gamma(\cdot)$	=	gamma function
γ	=	orthotropic parameter, $\sqrt{(k_z/k_r)}$, $\sqrt{(k_z/k_{xy})}$
δ_n	=	eigenvalues, $(n\pi/b, m\pi/c)$
δ_{xm}	=	eigenvalues, $\lambda_{xm} c$
δ_{yn}	=	eigenvalues, $\lambda_{yn} d$
ϵ	=	relative contact size, identical to a/b
ζ	=	dummy variable, m^{-1}
θ	=	temperature excess, $T - T_f$, K
$\bar{\theta}$	=	mean temperature excess, $\bar{T} - T_f$, K
κ	=	relative conductivity, k_2/k_1
λ_m	=	eigenvalues, $(m\pi/a, n\pi/d)$
μ	=	flux distribution parameter
ξ	=	transform variable, $z/\sqrt{(k_{\text{tp}}/k_{\text{ip}})}$
ϱ	=	equation parameter, $(\zeta + h/k_2)/(\zeta - h/k_2)$
ρ_1	=	radii ratio, a/b
ρ_2	=	radii ratio, b/c
τ	=	relative thickness, $t/\mathcal{L}$
ϕ, φ	=	spreading resistance functions
Ψ	=	spreading parameter, $k_2 R_s L$
ψ	=	spreading parameter, $4kaR_s$
	=	angular measurement, rad

Subscripts

e	=	edge
i	=	index denoting layers 1 and 2
ip	=	in-plane
m, n	=	mth and nth terms
r	=	r direction
tp	=	through plane
x	=	x direction
xy	=	xy plane
y	=	y direction
z	=	z direction

Introduction

THERMAL spreading resistance theory finds widespread application in electronics cooling, both at the board and chip level and in heat sink applications. It also arises in the thermal analysis of bolted joints and other mechanical connections resulting in discrete points of contact. Recently, a comprehensive review of the theory and application of thermal spreading resistances was undertaken by Yovanovich[1] and Yovanovich and Marotta.[2] Since these reviews, a number of new solutions and applications of spreading resistance theory have been addressed. These include, but are not limited to,

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the prediction of thermal resistance of electronic devices known as ball grid arrays,[3] the effect of heat source eccentricity,[4] the effect of heat spreaders in compound systems,[5–8] the effect of orthotropic properties,[9,10] and the issues of contact shape and edge cooling.[11–13]

This paper has two objectives. First, a general review of thermal spreading resistance theory in compound and orthotropic systems is undertaken, and solutions are reported for a number of useful compound and isotropic systems. Second, new solutions are developed for orthotropic systems using results for isotropic systems.

Presently, only a few analyses have been undertaken for orthotropic systems.[9,10] These solutions have only been presented for the circular disk and rectangular strip. It will be shown that with the appropriate transformations, solutions for isotropic systems may be applied to orthotropic systems with little effort.

Thermal Spreading Resistance

Thermal spreading resistance arises in multidimensional applications where heat enters a domain through a finite area. The total thermal resistance of the system may be defined as

$$R_T = (\bar{T}_s - T_f)/Q = \bar{\theta}_s/Q \tag{1}$$

where the mean source temperature is given by

$$\bar{\theta}_s = \frac{1}{A_s}\iint_{A_s} \theta(x, y, 0)\, \mathrm{d}A_s \tag{2}$$

In systems with adiabatic edges, the total thermal resistance is composed of two terms: a uniform flow or one-dimensional resistance and a spreading or multidimensional resistance that vanishes as the source area approaches the substrate area. These two components are combined as follows:

$$R_T = R_{1D} + R_s \tag{3}$$

where

$$R_{1D} = \sum_{i=1}^{N} \frac{t_i}{k_i A} + \frac{1}{hA} \tag{4}$$

In this paper, we are concerned mainly with the second term, R_s, the spreading resistance.

Thermal spreading resistance analysis requires the solution of Laplace's equation in either two or three dimensions. For an isotropic system, Laplace's equation takes the form

$$\nabla^2 T = 0 \tag{5}$$

or for an orthotropic system, it has the form

$$\nabla \cdot (k\nabla T) = 0 \tag{6}$$

where k has a unique value in each of the three principal coordinate directions.

In most applications, the following boundary conditions are applied:

$$\frac{\partial T}{\partial n} = 0, \qquad n = x, y, \quad \text{or} \quad r \tag{7}$$

along the adiabatic edges and at the centroid of the disk or channel, or in the case of edge cooling,

$$\frac{\partial T}{\partial n} + \frac{h_e}{k}(T - T_f) = 0, \qquad n = x, y, \quad \text{or} \quad r \tag{8}$$

is applied along the edges in place of Eq. (7).

On the upper and lower surfaces, the following conditions are applied:

$$\left.\begin{aligned} \frac{\partial T}{\partial z} &= 0, & &A \text{ outside } A_s \\ \frac{\partial T}{\partial z} &= -\frac{q}{k_z}, & &A \text{ inside } A_s \end{aligned}\right\} z = 0 \tag{9}$$

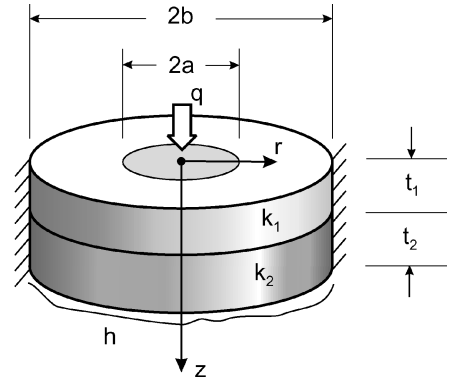

Fig. 1 Compound flux tube with circular heat source.

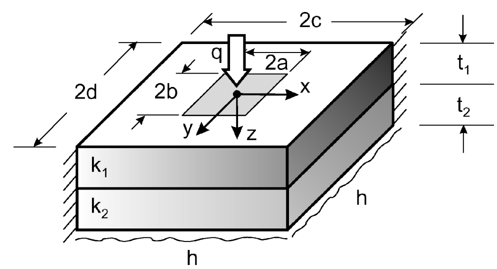

Fig. 2 Compound flux channel with rectangular heat source.

where A_s is the area of the heat source and

$$\frac{\partial T}{\partial z} + \frac{h}{k_z}(T - T_f) = 0, \qquad z = t \tag{10}$$

on the lower surface.

In compound systems (Figs. 1 and 2) Laplace's equation must be written for each layer in the system,

$$\nabla^2 T_i = 0, \qquad i = 1, 2 \tag{11}$$

The boundary conditions now become

$$\frac{\partial T_i}{\partial n} = 0, \qquad n = x, y, \quad \text{or} \quad r \tag{12}$$

along the edges and at the centroid of the disk or channel, whereas continuity of temperature and heat flux at the interface is required, yielding two additional boundary conditions:

$$\left.\begin{aligned} T_1 &= T_2 \\ k_1\frac{\partial T_1}{\partial z} &= k_2\frac{\partial T_2}{\partial z} \end{aligned}\right\}, \qquad z = t_1 \tag{13}$$

Finally, along the upper and lower surfaces, the following conditions must be applied:

$$\left.\begin{aligned} \frac{\partial T_1}{\partial z} &= 0, & &A \text{ outside } A_s \\ \frac{\partial T_1}{\partial z} &= -\frac{q}{k_z}, & &A \text{ inside } A_s \end{aligned}\right\} z = 0 \tag{14}$$

$$\frac{\partial T_2}{\partial z} + \frac{h}{k_z}(T_2 - T_f) = 0, \qquad z = t_1 + t_2 \tag{15}$$

Because of the nature of the solution procedure, the total thermal resistance may be analyzed as two problems. One is steady one-dimensional conduction, which yields the uniform flow component of the thermal resistance, whereas the other is a multidimensional conduction analysis using Fourier series or integral transform methods to solve an eigenvalue problem.[14,15] This paper is mainly concerned with the solution to the thermal spreading resistance component in systems with one or two layers. In cases where edge cooling is present, the system is always multidimensional, and therefore, the thermal resistance represents the total resistance.

Compound and Isotropic Systems

A review of thermal spreading resistance in Cartesian and cylindrical systems is given for compound flux tubes, channels, and annular sectors (Figs. 1–6). These solutions contain many special cases involving spreading resistance in isotropic disks and channels, flux tubes and channels, and half spaces. Since the publication of Ref. 1, a number of new solutions for spreading resistance in rectangular flux channels have been obtained for many special cases.[4,5,12] In addition, the solution for an annular sector was recently obtained by the authors.[8] Because many of these general results will be required later, they are reported here for rectangular and cylindrical systems. The effect of edge cooling was recently examined by Yovanovich[13] for the isotropic flux tube (Fig. 4) and by Muzychka et al.[12] for the isotropic flux channel (Fig. 5). These results are also given next. With additional modification, they may also be applied to orthotropic systems.

Flux Tubes

Thermal spreading resistance solutions in isotropic and compound disks (Fig. 1), flux channels, and half-spaces are presented by Yovanovich et al.[6,7] A general solution for the compound disk was first obtained by Yovanovich et al.[7] The general solution for the dimensionless spreading resistance $\psi = 4k_1 a R_s$ is

$$\psi = \frac{8}{\pi\epsilon}\sum_{n=1}^{\infty} A_n(n,\epsilon)B_n(n,\tau,\tau_1)\frac{J_1(\delta_n\epsilon)}{\delta_n\epsilon} \qquad (16)$$

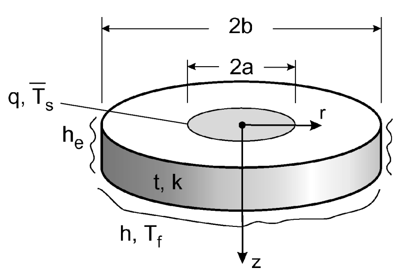

Fig. 3 Isotropic flux tube with circular heat source and edge cooling.

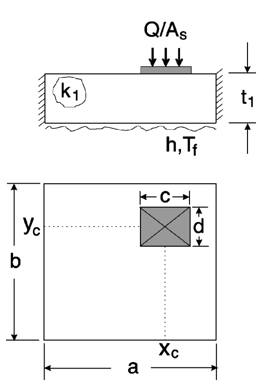

Fig. 4 Flux channel with eccentric heat source.

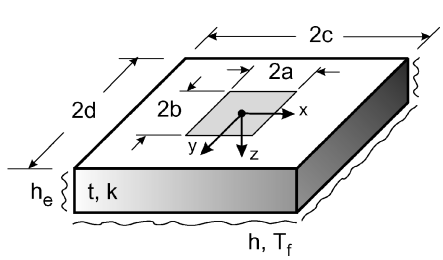

Fig. 5 Isotropic flux channel with rectangular heat source and edge cooling.

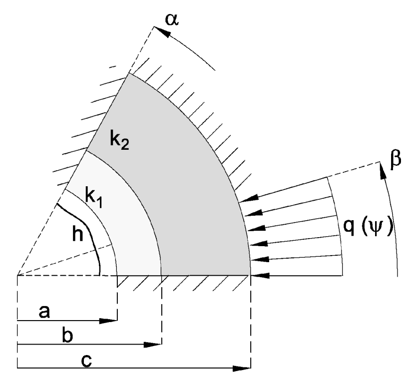

Fig. 6 Compound annular sector.

where

$$A_n = -\frac{2\epsilon J_1(\delta_n\epsilon)}{\delta_n^2 J_0^2(\delta_n)}, \qquad B_n = \frac{\phi_n \tanh(\delta_n) - \varphi_n}{(1-\phi_n)} \qquad (17)$$

The functions ϕ_n and φ_n are defined as follows:

$$\phi_n = [(\kappa - 1)/\kappa]\cosh(\delta_n\tau_1)[\cosh(\delta_n\tau_1) - \varphi_n \sinh(\delta_n\tau_1)] \qquad (18)$$

where

$$\varphi_n = \frac{\delta_n + Bi\tanh(\delta_n\tau)}{\delta_n\tanh(\delta_n\tau) + Bi} \qquad (19)$$

The eigenvalues δ_n are solutions to $J_1(\delta_n) = 0$, $Bi = hb/k_2$, $\tau = t/b$, and $\tau_1 = t_1/b$. The general solution given earlier reduces to the case of an isotropic disk when $\kappa = 1$.

Yovanovich[13] obtained the following result for an edge cooled isotropic flux tube:

$$\psi = \frac{16}{\pi\epsilon}\sum_{n=1}^{\infty}\left(\frac{2}{\delta_n\epsilon}\right)^{\mu}\frac{\Gamma(2+\mu)J_{1+\mu}(\delta_n\epsilon)J_1(\delta_n\epsilon)\varphi_n}{\delta_n^3\left[J_0^2(\delta_n) + J_1^2(\delta_n)\right]} \qquad (20)$$

where φ_n is given by Eq. (19) and $\psi = 4akR_T$, $\tau = t/b$, $Bi = hb/k$, and $\epsilon = a/b$, and δ_n are the eigenvalues. The eigenvalues are obtained from application of the boundary condition along the disk edges and require numerical solution to the following transcendental equation:

$$\delta_n J_1(\delta_n) = Bi_e J_0(\delta_n) \qquad (21)$$

where $\delta_n = \lambda_n b$ and $Bi_e = h_e b/k$ is the edge Biot number. A unique set of eigenvalues is computed for each value of Bi_e. Simplified expressions for predicting the eigenvalues were developed by Yovanovich[13] using the Newton–Raphson method. The solution reported earlier is valid for any value of the flux parameter $\mu > -1$. However, only values of $\mu = -\frac{1}{2}$, $\mu = 0$, and $\mu = \frac{1}{2}$ have practical usage.

Flux Channels

Thermal spreading resistances in rectangular flux channels have recently been examined by the authors.[4,5,12] Yovanovich et al.[5] obtained a solution for a compound rectangular flux tube having a central heat source (Fig. 2). This general solution also simplifies for many cases of semi-infinite flux channels and half-space solutions. More recently, the authors[4] developed a solution for a single eccentric heat source on compound and isotropic flux channels (Fig. 4). The results of Muzychka et al.[4] were also extended to systems having multiple arbitrarily placed heat sources. Finally, Muzychka et al.[12] obtained a solution for a rectangular flux channel with edge cooling (Fig. 5).

The spreading resistance of Yovanovich et al.[5] for a compound flux channel is determined from the following general expression according to the notation in Fig. 2:

$$R_s = \frac{1}{2a^2cdk_1}\sum_{m=1}^{\infty}\frac{\sin^2(a\delta_m)}{\delta_m^3}\cdot\varphi(\delta_m) + \frac{1}{2b^2cdk_1}\sum_{n=1}^{\infty}\frac{\sin^2(b\lambda_n)}{\lambda_n^3}\cdot\varphi(\lambda_n) + \frac{1}{a^2b^2cdk_1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin^2(a\delta_m)\sin^2(b\lambda_n)}{\delta_m^2\lambda_n^2\beta_{mn}}\cdot\varphi(\beta_{mn}) \tag{22}$$

where

$$\varphi(\zeta) = \frac{(\alpha e^{4\zeta t_1} + e^{2\zeta t_1}) + \varrho\left(e^{2\zeta(2t_1+t_2)} + \alpha e^{2\zeta(t_1+t_2)}\right)}{(\alpha e^{4\zeta t_1} - e^{2\zeta t_1}) + \varrho\left(e^{2\zeta(2t_1+t_2)} - \alpha e^{2\zeta(t_1+t_2)}\right)} \tag{23}$$

$$\varrho = \frac{\zeta + h/k_2}{\zeta - h/k_2}, \qquad \alpha = \frac{1-\kappa}{1+\kappa}$$

with $\kappa = k_2/k_1$. The eigenvalues for these solutions are $\delta_m = m\pi/c$, $\lambda_n = n\pi/d$, and $\beta_{mn} = \sqrt{(\delta_m^2 + \lambda_n^2)}$. The given general solution reduces to the case of an isotropic channel when $\kappa = 1$.

The general solution for the mean temperature excess of a single eccentric heat source was obtained by Muzychka et al.[4] The thermal spreading resistance for an arbitrarily located heat source using the notation of Fig. 4 is

$$R_s = \frac{2}{k_1abc^2}\sum_{m=1}^{\infty}A_m\frac{\cos(\lambda_m X_c)\sin\left(\frac{1}{2}\lambda_m c\right)}{\lambda_m}\cdot\varphi(\lambda_m) \times \frac{2}{k_1abd^2}\sum_{n=1}^{\infty}A_n\frac{\cos(\delta_n Y_c)\sin\left(\frac{1}{2}\delta_n d\right)}{\delta_n}\cdot\varphi(\delta_n) + \frac{4}{k_1abc^2d^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn} \times \frac{\cos(\delta_n Y_c)\sin\left(\frac{1}{2}\delta_n d\right)\cos(\lambda_m X_c)\sin\left(\frac{1}{2}\lambda_m c\right)}{\lambda_m\delta_n}\cdot\varphi(\beta_{mn}) \tag{24}$$

where

$$A_m = \frac{2\{\sin([(2X_c + c)/2]\lambda_m) - \sin([(2X_c - c)/2]\lambda_m)\}}{\lambda_m^2}$$

$$A_n = \frac{2\{\sin([(2Y_c + d)/2]\delta_n) - \sin([(2Y_c - d)/2]\delta_n)\}}{\delta_n^2}$$

$$A_{mn} = \frac{16\cos(\lambda_m X_c)\sin\left(\frac{1}{2}\lambda_m c\right)\cos(\delta_n Y_c)\sin\left(\frac{1}{2}\delta_n d\right)}{\beta_{mn}\lambda_m\delta_n} \tag{25}$$

and where $\kappa = k_2/k_1$ and ζ is replaced by λ_m, δ_n, or β_{mn}, accordingly in Eq. (23). The eigenvalues for these solutions are $\lambda_m = m\pi/a$, $\delta_n = n\pi/b$, and $\beta_{mn} = \sqrt{(\lambda_m^2 + \delta_n^2)}$.

The preceding solution may be used to calculate the thermal spreading resistance for a source located at any point on a compound or isotropic rectangular flux channel. Details of the local surface temperature distribution may be computed from additional expressions presented by Muzychka et al.[4] for single or multiple heat sources. The general solution given earlier reduces to the case of an isotropic channel when $\kappa = 1$.

Muzychka et al.[12] obtained the following result for the total system resistance for an edge cooled isotropic flux channel:

$$R_T = \frac{cd}{ka^2b^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \times \frac{\sin^2(\delta_{xm}a/c)\sin^2(\delta_{yn}b/d)\phi_{mn}}{\delta_{xm}\delta_{yn}\beta_{mn}[\sin(2\delta_{xm})/2 + \delta_{xm}][\sin(2\delta_{yn})/2 + \delta_{yn}]} \tag{26}$$

where

$$\phi_{mn} = \frac{t\beta_{mn} + (ht/k)\tanh(\beta_{mn}t)}{(ht/k) + t\beta_{mn}\tanh(\beta_{mn}t)} \tag{27}$$

The eigenvalues are obtained from the following equations:

$$\delta_{xm}\sin(\delta_{xm}) = Bi_{e,x}\cos(\delta_{xm})$$

$$\delta_{yn}\sin(\delta_{yn}) = Bi_{e,y}\cos(\delta_{yn}) \tag{28}$$

where $Bi_{e,x} = h_{e,x}c/k$, $\delta_{xm} = \lambda_{xm}c$, $Bi_{e,y} = h_{e,y}d/k$, and $\delta_{yn} = \lambda_{yn}d$. The edge cooling coefficients $h_{e,x}$ and $h_{e,y}$, need not be equal. These equations must be solved numerically for a finite number of eigenvalues for each specified value of the edge cooling Biot numbers. The separation constant β_{mn} is now defined as

$$\beta_{mn} = \sqrt{(\delta_{xm}/c)^2 + (\delta_{yn}/d)^2} \tag{29}$$

Annular Sectors

Finally, Muzychka et al.[14] obtained the solution for a compound annular sector with arbitrary flux distribution. The general solution is valid for any heat flux distribution defined by

$$q(\psi) = K[1 - (\psi/\beta)^2]^{\mu}, \qquad 0 \le \psi < \beta \tag{30}$$

where

$$K = \frac{Q}{\beta c}\frac{2}{\sqrt{\pi}}\frac{\Gamma(\mu + 3/2)}{\Gamma(\mu + 1)} \tag{31}$$

with $\mu > -1$. However, only three cases are of practical interest. These are the uniform flux ($\mu = 0$), parabolic flux ($\mu = \frac{1}{2}$), and inverted parabolic flux ($\mu = -\frac{1}{2}$). The inverted parabolic flux distribution is representative of the isothermal boundary condition for values of $\epsilon = \beta/\alpha < 0.5$. The final general result for the spreading resistance is

$$\Psi = \frac{2}{\pi^2\epsilon}\Gamma\left(\mu + \frac{3}{2}\right)\sum_{n=1}^{\infty}\left(\frac{2}{n\pi\epsilon}\right)^{\mu+\frac{1}{2}}\frac{\sin(n\pi\epsilon)}{n^2}J_{\mu+\frac{1}{2}}(n\pi\epsilon)\varphi_n \tag{32}$$

where $\Psi = R_s k_2 L$, $\epsilon = \beta/\alpha$, and the parameter φ_n determines the effect of shell thicknesses, layer conductivities, and heat transfer coefficient. It is defined as

$$\varphi_n = \left[\frac{(F_1 Bi + F_2\lambda_n)\kappa + (F_3 Bi + F_4\lambda_n)}{(F_4 Bi + F_3\lambda_n)\kappa + (F_2 Bi + F_1\lambda_n)}\right] \tag{33}$$

where

$$F_1 = 1 - (\rho_1)^{2\lambda_n} + (\rho_2)^{2\lambda_n} - (\rho_1\rho_2)^{2\lambda_n}$$

$$F_2 = 1 + (\rho_1)^{2\lambda_n} + (\rho_2)^{2\lambda_n} + (\rho_1\rho_2)^{2\lambda_n}$$

$$F_3 = 1 + (\rho_1)^{2\lambda_n} - (\rho_2)^{2\lambda_n} - (\rho_1\rho_2)^{2\lambda_n}$$

$$F_4 = 1 - (\rho_1)^{2\lambda_n} - (\rho_2)^{2\lambda_n} + (\rho_1\rho_2)^{2\lambda_n} \tag{34}$$

The eigenvalues are $\lambda_n = n\pi/\alpha$. The total dimensionless thermal resistance for the basic element shown in Fig. 6 is

$$R_T^* = \Psi + \frac{2\pi}{\alpha}R_{1D}^* \tag{35}$$

where

$$R_{1D}^* = \kappa\frac{\ln(1/\rho_1)}{2\pi} + \frac{\ln(1/\rho_2)}{2\pi} + \frac{\kappa}{2\pi Bi} \tag{36}$$

and where $R^* = k_2RL$, $0 < \rho_1 = a/b < 1$, $0 < \rho_2 = b/c < 1$, $\kappa = k_2/k_1$, and $Bi = ha/k_1$.

When $\kappa = 1$, an isotropic annular sector is obtained.

This concludes the review of solutions for isotropic and compound systems. In addition to these new solutions, the effect of shape was recently examined by the authors. Muzychka et al.[11] showed that the thermal spreading resistance was a weak function of shape and geometry. Equivalency was established between the flux tube and the flux channel. We now proceed to develop the necessary relationships to address the issue of orthotropic properties.

Application to Orthotropic Systems

We now examine the issue of computing thermal spreading resistance in orthotropic flux tubes and flux channels (Figs. 7 and 8). Laplace's equation, Eq. (6), may be expanded for an orthotropic system and written as

$$k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} + k_z \frac{\partial^2 T}{\partial z^2} = 0 \tag{37}$$

in Cartesian coordinates, or

$$k_r \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} \right) + k_z \frac{\partial^2 T}{\partial z^2} = 0 \tag{38}$$

in cylindrical coordinates.

Equations (37) and (38) may be transformed such that the governing equation and boundary conditions are reduced to those for an equivalent isotropic system. The solution for the orthotropic system is then easily obtained from this equivalent system. A transformation[16] for composite orthotropic materials was proposed for thermal spreading resistance of a circular heat source on a half-space. This transformation will be applied to isotropic spreading resistance solutions for finite circular and rectangular disks. Further discussion on the transformation of orthotropic systems to isotropic systems may be found by Carslaw and Jaeger[14] and Ozisik.[15]

Results will be presented for two cases: the orthotropic disk, that is, $k_r \neq k_z$, and the orthotropic channel, where the in-plane and through-plane conductivities are different, that is, $k_x = k_y \neq k_z$. In orthotropic systems such as printed circuit boards, series and parallel models are often used to define k_r, k_{xy}, and k_z. These are defined as[16,17]

$$k_r, k_{xy} = \frac{\sum_{i=1}^{N} k_i t_i}{t} \tag{39}$$

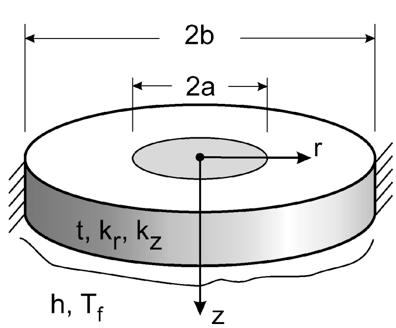

Fig. 7 Orthotropic flux tube with circular heat source.

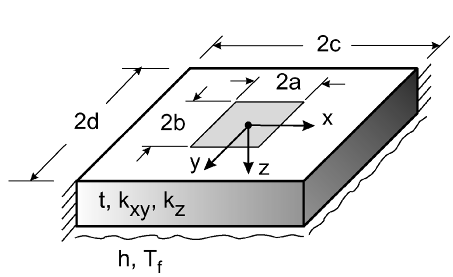

Fig. 8 Orthotropic flux channel with rectangular heat source.

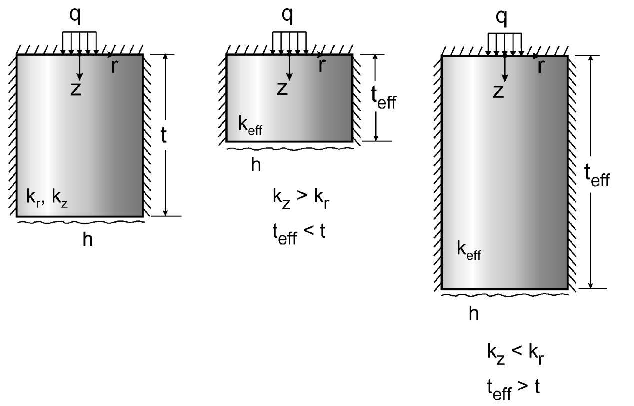

Fig. 9 Transformation of orthotropic system to an isotropic system.

and

$$k_z = t \bigg/ \sum_{i=1}^{N} \frac{t_i}{k_i} \tag{40}$$

After transformation, the effective flux tube or channel will have either increased or decreased in length as shown in Fig. 9, depending on the ratio of the through-plane conductivity k_z to the in-plane conductivity k_r, k_{xy}.

In general, thermal spreading resistance in multilayered systems is a strong function of the size and distribution of conducting layers. However, if the source size is considerably larger than the thickness of individual layers,[10] the preceding relations for determining effective series and parallel conductivities may be applied.

Finally, many other special cases that naturally arise from the general solutions, such as the half-space and semi-infinite flux tube, will also be discussed.

Flux Tubes

In a cylindrical orthotropic system (Fig. 7), Laplaces's equation, Eq. (38) may be transformed using $\xi = z/\gamma$, where $\gamma = \sqrt{(k_z/k_r)}$, and $\theta = T - T_f$, to yield

$$\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} + \frac{\partial^2 \theta}{\partial \xi^2} = 0 \tag{41}$$

which is subjected to the following transformed boundary conditions:

$$r = 0, b, \qquad \frac{\partial \theta}{\partial r} = 0$$

$$\xi = 0, \qquad \frac{\partial \theta}{\partial \xi} = -\frac{q}{k_{\text{eff}}}, \qquad 0 \leq r < a$$

$$\frac{\partial \theta}{\partial \xi} = 0, \qquad a < r \leq b$$

$$\xi = t_{\text{eff}}, \qquad \frac{\partial \theta}{\partial \xi} = -\frac{h}{k_{\text{eff}}}\theta \tag{42}$$

Equations (41) and (42) are now in the same form as that for an isotropic disk, except an effective thermal conductivity $k_{\text{eff}} = \sqrt{(k_r k_z)}$ now replaces the isotropic thermal conductivity and an effective thickness, $t_{\text{eff}} = t/\gamma$, now replaces the flux tube thickness. The solution for this case is[1]

$$\psi = 4k_{\text{eff}} a R_s = \frac{16}{\pi\epsilon} \sum_{n=1}^{\infty} \frac{J_1^2(\delta_n \epsilon)}{\delta_n^3 J_0^2(\delta_n)} \varphi_n \tag{43}$$

where

$$\varphi_n = \frac{\delta_n + Bi_{\text{eff}} \tanh(\delta_n \tau_{\text{eff}})}{\delta_n \tanh(\delta_n \tau_{\text{eff}}) + Bi_{\text{eff}}} \tag{44}$$

The dimensionless thickness and Biot number now become $\tau_{\text{eff}} = t_{\text{eff}}/b$ and $Bi_{\text{eff}} = hb/k_{\text{eff}}$, where the effective conductivity is $k_{\text{eff}} = \sqrt{(k_r k_z)}$, effective thickness is $t_{\text{eff}} = t/\gamma$, and $\epsilon = a/b$. The eigenvalues δ_n are obtained from $J_1(\delta_n) = 0$.

In the case of edge cooling, the solution of Yovanovich[13] becomes

$$\psi = 4ak_{\text{eff}}R_T = \frac{16}{\pi\epsilon}\sum_{n=1}^{\infty}\left(\frac{2}{\delta_n\epsilon}\right)^{\mu}\frac{\Gamma(2+\mu)J_{1+\mu}(\delta_n\epsilon)J_1(\delta_n\epsilon)\varphi_n}{\delta_n^3\left[J_0^2(\delta_n)+J_1^2(\delta_n)\right]} \tag{45}$$

where φ_n is given by Eq. (44), $\tau_{\text{eff}} = t_{\text{eff}}/b$, $Bi_{\text{eff}} = hb/k_{\text{eff}}$, and $\epsilon = a/b$, and δ_n are the eigenvalues. The eigenvalues are now obtained from

$$\delta_n J_1(\delta_n) = (h_e b/k_r)J_0(\delta_n) \tag{46}$$

or, after transforming for consistency,

$$\delta_n J_1(\delta_n) = Bi_e\gamma J_0(\delta_n) \tag{47}$$

where $Bi_e = h_e b/k_{\text{eff}}$.

Flux Channels

In a rectangular orthotropic system (Fig. 8), with $k_x = k_y = k_{xy}$, Laplaces's equation (37) may also be transformed using $\xi = z/\gamma$, where $\gamma = \sqrt{(k_z/k_{xy})}$ and $\theta = T - T_f$, to yield

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial \xi^2} = 0 \tag{48}$$

which is subjected to the following transformed boundary conditions:

$$x = 0, c, \quad \frac{\partial\theta}{\partial x} = 0, \qquad y = 0, d, \quad \frac{\partial\theta}{\partial y} = 0$$

$$\xi = 0, \quad \frac{\partial\theta}{\partial\xi} = -\frac{q}{k_{\text{eff}}}, \quad 0 \le x < a, \quad 0 \le y < b$$

$$\frac{\partial\theta}{\partial\xi} = 0, \quad a < x \le c, \quad b < y \le d$$

$$\xi = t_{\text{eff}}, \quad \frac{\partial\theta}{\partial\xi} = -\frac{h}{k_{\text{eff}}}\theta \tag{49}$$

Equations (48) and (49) are now in the same form as those for an isotropic flux tube, except an effective thermal conductivity $k_{\text{eff}} = \sqrt{(k_{xy}k_z)}$ now replaces the isotropic thermal conductivity and an effective thickness, $t_{\text{eff}} = t/\gamma$, replaces the flux channel thickness. The solution for this case is[5]

$$R_s = \frac{1}{2a^2cdk_{\text{eff}}}\sum_{m=1}^{\infty}\frac{\sin^2(a\delta_m)}{\delta_m^3}\cdot\varphi(\delta_m) + \frac{1}{2b^2cdk_{\text{eff}}}\sum_{n=1}^{\infty}\frac{\sin^2(b\lambda_n)}{\lambda_n^3}\cdot\varphi(\lambda_n) + \frac{1}{a^2b^2cdk_{\text{eff}}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin^2(a\delta_m)\sin^2(b\lambda_n)}{\delta_m^2\lambda_n^2\beta_{m,n}}\cdot\varphi(\beta_{m,n}) \tag{50}$$

where

$$\varphi(\zeta) = \frac{(e^{2\zeta t_{\text{eff}}}+1)\zeta t_{\text{eff}} - (1-e^{2\zeta t_{\text{eff}}})ht_{\text{eff}}/k_{\text{eff}}}{(e^{2\zeta t_{\text{eff}}}-1)\zeta t_{\text{eff}} + (1+e^{2\zeta t_{\text{eff}}})ht_{\text{eff}}/k_{\text{eff}}} \tag{51}$$

and ζ is a dummy variable denoting the respective eigenvalues. The eigenvalues for these solutions are $\delta_m = m\pi/c$, $\lambda_n = n\pi/d$, and $\beta_{m,n} = \sqrt{(\delta_m^2+\lambda_n^2)}$.

The thermal spreading resistance for an arbitrarily located heat source using the notation of Fig. 4 is written as

$$R_s = \frac{2}{k_{\text{eff}}abc^2}\sum_{m=1}^{\infty}A_m\frac{\cos(\lambda_m X_c)\sin\left(\frac{1}{2}\lambda_m c\right)}{\lambda_m}\cdot\varphi(\lambda_m) + \frac{2}{k_{\text{eff}}abd^2}\sum_{n=1}^{\infty}A_n\frac{\cos(\delta_n Y_c)\sin\left(\frac{1}{2}\delta_n d\right)}{\delta_n}\cdot\varphi(\delta_n) + \frac{4}{k_{\text{eff}}abc^2d^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn} \times\frac{\cos(\delta_n Y_c)\sin\left(\frac{1}{2}\delta_n d\right)\cos(\lambda_m X_c)\sin\left(\frac{1}{2}\lambda_m c\right)}{\lambda_m\delta_n}\cdot\varphi(\beta_{mn}) \tag{52}$$

where $\varphi(\zeta)$ is also given by Eq. (51), with ζ being replaced by λ_m, δ_n, or β_{mn}, accordingly. The eigenvalues for this solution are $\lambda_m = m\pi/a$, $\delta_n = n\pi/b$, and $\beta_{mn} = \sqrt{(\lambda_m^2+\delta_n^2)}$.

The preceding result may now be used to calculate spreading resistances in electronic circuit boards and other systems having orthotropic characteristics. Furthermore, if desired, the general results from Muzychka et al.[4] may be applied for multiple heat sources placed on an orthotropic media.

Finally, the solution of Muzychka et al.[12] for the total system resistance for an edge cooled isotropic flux channel may now be transformed for an orthotropic channel:

$$R_T = \frac{cd}{k_{\text{eff}}a^2b^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \times\frac{\sin^2(\delta_{xm}a/c)\sin^2(\delta_{yn}b/d)\phi_{mn}}{\delta_{xm}\delta_{yn}\beta_{mn}[\sin(2\delta_{xm})/2+\delta_{xm}][\sin(2\delta_{yn})/2+\delta_{yn}]} \tag{53}$$

where

$$\phi_{mn} = \frac{t_{\text{eff}}\beta_{mn} + (ht_{\text{eff}}/k_{\text{eff}})\tanh(\beta_{mn}t_{\text{eff}})}{(ht_{\text{eff}}/k_{\text{eff}}) + t_{\text{eff}}\beta_{mn}\tanh(\beta_{mn}t_{\text{eff}})} \tag{54}$$

The eigenvalues are obtained from the following equations:

$$\delta_{xm}\sin(\delta_{xm}) = (h_{e,x}c/k_{xy})\cos(\delta_{xm})$$
$$\delta_{yn}\sin(\delta_{yn}) = (h_{e,y}d/k_{xy})\cos(\delta_{yn}) \tag{55}$$

or for consistency, we may write

$$\delta_{xm}\sin(\delta_{xm}) = Bi_{e,x}\gamma\cos(\delta_{xm})$$
$$\delta_{yn}\sin(\delta_{yn}) = Bi_{e,y}\gamma\cos(\delta_{yn}) \tag{56}$$

where $Bi_{e,x} = h_{e,x}c/k_{\text{eff}}$, $\delta_{xm} = \lambda_{xm}c$, $Bi_{e,y} = h_{e,y}d/k_{\text{eff}}$, and $\delta_{yn} = \lambda_{yn}d$. Finally, the separation constant β_{mn} is defined by Eq. (29).

Special Cases

The general solutions given by Eqs. (43), (45), (50), (52), and (53) contain many special limits. These include semi-infinite flux tubes, channels, and half-space limits. These and other special cases are discussed by Yovanovich,[1] Muzychka et al.,[4,11,12] Yovanovich et al.,[5–7] and Yovanovich.[13]

Conclusions

A review of thermal spreading resistance solutions for compound and isotropic systems with and without edge cooling was given. By means of simple transformations, all of the solutions for thermal spreading resistance in isotropic flux tubes and channels were applied to orthotropic systems. A number of special cases involving orthotropic systems were discussed. The effects of edge cooling, source eccentricity, and heat flux distribution were also examined.

It was shown that orthotropic spreading resistance solutions can be obtained by applying the following transformation rules:

$$k \rightarrow k_{\mathrm{eff}} = \sqrt{k_{\mathrm{ip}} k_{\mathrm{tp}}} \tag{57}$$

where k_{ip} and k_{tp} are the in-plane and through-plane thermal conductivity, and

$$t \rightarrow t_{\mathrm{eff}} = t/\gamma \tag{58}$$

where $\gamma = \sqrt{(k_{\mathrm{tp}}/k_{\mathrm{ip}})}$.

These rules may also be applied to many of the solutions reported by Yovanovich[1] and Yovanovich and Marotta[2] for systems defined by r, z or x, y, z, where heat enters through the $z = 0$ plane.

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