Laminar Flow Friction and Heat Transfer in Non-Circular Ducts and Channels Part II- Thermal Problem

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LAMINAR FLOW FRICTION AND HEAT TRANSFER IN NON-CIRCULAR DUCTS AND CHANNELS PART II - THERMAL PROBLEM

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ABSTRACT

A detailed review and analysis of the thermal characteristics of laminar developing and fully developed flow in non-circular ducts is presented. New models are proposed which simplify the prediction of Nusselt numbers for three fundamental flows: the combined entrance problem, the Graetz problem, and thermally fully developed flow in most non-circular duct geometries found in heat exchanger applications. By means of scaling analysis it is shown that the complete problem may be easily analyzed by combining the asymptotic results for short and long ducts. By means of a new characteristic length scale, the square root of cross-sectional area, the effect of duct shape has been reduced. The new model has an accuracy of ± 10 percent, or better, for most common duct shapes. Both singly and doubly connected ducts are considered.

NOMENCLATURE

A	=	flow area, m^2
a, b	=	major and minor axes of ellipse or rectangle, m
B_i	=	constants, $i = 1 \ldots 4$
C	=	constant
C_i	=	constants, $i = 1 \ldots 5$
c_p	=	specific heat, $J/kg\,K$
D_h	=	hydraulic diameter of plain channel, $\equiv 4A/P$
f	=	friction factor $\equiv \tau/(\frac{1}{2}\rho U^2)$
h	=	heat transfer coefficient, W/m^2K
k	=	thermal conductivity, W/mK
L	=	duct length, m
L^*	=	dimensionless thermal length,$\equiv L/\mathcal{L}Re_{\mathcal{L}}Pr$
$\mathcal{L}$	=	characteristic length scale, m
m	=	correlation parameter
$\dot{m}$	=	mass flow rate, kg/s
n	=	inward directed normal
$Nu_{\mathcal{L}}$	=	Nusselt number, $\equiv h\mathcal{L}/k$
P	=	perimeter, m
Pr	=	Prandtl number, $\equiv \nu/\alpha$
q	=	heat flux, W/m^2
r	=	radius, m
$Re_{\mathcal{L}}$	=	Reynolds number, $\equiv U\mathcal{L}/\nu$
s	=	arc length, m
T	=	temperature, K
u, v, w	=	velocity components, m/s
U	=	average velocity, m/s
x, y, z	=	cartesian coordinates, m
Y	=	dimensionless coordinate, $\equiv y/\mathcal{L}$
z	=	axial coordinate, m
Z	=	dimensionless position for thermally developing flows, $\equiv z/\mathcal{L}Re_{\mathcal{L}}Pr$

Greek Symbols

α	=	thermal diffusivity, m^2/s
δ	=	hydrodynamic boundary layer thickness, m
Δ	=	thermal boundary layer thickness, m
ϵ	=	aspect ratio, $\equiv b/a$
γ	=	symmetry parameter
μ	=	dynamic viscosity, Ns/m^2
ν	=	kinematic viscosity, m^2/s
ρ	=	fluid density, kg/m^3
τ	=	wall shear stress, N/m^2
θ	=	temperature excess, $T - T_o$, K

Subscripts

$\sqrt{A}$	=	based on the square root of area
D_h	=	based on the hydraulic diameter
H	=	based on isoflux condition
h	=	hydrodynamic
$\mathcal{L}$	=	based on the arbitrary length $\mathcal{L}$
m	=	mixed or bulk value
T	=	based on isothermal condition
t	=	thermal

Superscripts

$\overline{(\cdot)}$	=	denotes average value of $(\cdot)$

INTRODUCTION

Heat transfer in non-circular ducts of constant cross-sectional area is of particular interest in the design of compact heat exchangers. In these applications passages are generally short and usually composed of cross-sections such as triangular or rectangular geometries in addition to the circular tube or parallel plate channel. Also, due to the wide range of applications, fluid Prandtl numbers usually vary between $0.1 < Pr < \infty$, which covers a wide range of fluids encompassing gases and highly viscous liquids such as automotive oils. In the second part of this paper, the authors develop models for thermally fully developed flow, thermally developing or Graetz flow, and simultaneously developing flow in circular and non-circular ducts and channels.

A review of the literature [1,2] reveals that the only models available for predicting heat transfer in the combined entry region are those of Churchill and Ozoe [3,4] for the circular duct and Stephan [5,6] for the parallel plate channel and circular duct. Recently, Garimella et al. [7] developed empirical expressions for the rectangular channel, while numerical data for polygonal ducts were obtained by Asako et al. [8]. Additional data for the rectangular, circular, triangular, and parallel plate channel are available in Shah and London [1], Rohsenow et al. [9], and Kakac and Yener [10]. Yilmaz and Cihan [11,12] provide a complex correlation method for the case of thermally developing flow with fully developed hydrodynamic flow, often referred to as the Graetz problem or Graetz flow. More recently, the authors have presented simpler models for both the Graetz problem [13], the combined entrance problem [14], and natural convection in vertical isothermal ducts [15]. Additional results are reported in Muzychka [16].

In Part I of this paper, the hydrodynamic problem was considered in detail. Part II of this paper will present new models for the associated thermal problem. These models have been developed using the Churchill and Usagi [17] asymptotic correlation method, as was done in Part I for the hydrodynamic problem. In this paper, the asymptotic solutions for thermally fully developed flow, $L >> L_h, L >> L_t$, thermally developing flow, $L >> L_h, L << L_t$, and the combined entry problem, $L << L_h, L << L_t$, are used to develop a more general model for predicting heat transfer coefficients in non-circular ducts. Here L_t denotes the thermal entry length and L_h the hydrodynamic entrance length.

GOVERNING EQUATIONS

The governing equations for steady, constant property, incompressible flow, in the thermal entrance region in a non-circular duct or channel are:

$$\nabla \cdot \vec{V} = 0 \tag{1}$$

$$\rho \vec{V} \cdot \nabla \vec{V} = -\nabla p + \mu \nabla^2 \vec{V} \tag{2}$$

$$\rho c_p \vec{V} \cdot \nabla T = k \nabla^2 T \tag{3}$$

Simultaneous solution of the continuity, Eq. (1), and momentum, Eq. (2), equations subject to the no slip condiction at the duct wall, $\vec{V} = 0$, the boundedness condition along the duct duct axis, $\vec{V} \neq \infty$, and a constant initial velocity, $\vec{V} = U\vec{k}$, are required to characterize the flow. In the energy equation, the flow is subject to constant inlet temperature, the boundedness condition along the duct axis, and either uniform wall temperature (UWT) usually denoted with a subscript (T) or uniform wall flux (UWF) condition, usually denoted with a subscript (H), at the duct wall.

In the next section, scaling analysis [18] is used to show the appropriate form of the solution for both short and long ducts. Later, asymptotic analysis [19] is used to develop a new model.

SCALE ANALYSIS

We now examine the energy equation and consider the various balances implied under particular flow conditions. The energy equation represents a balance between transverse conduction and axial convection, i.e.,

$$\underbrace{\rho c_p \vec{V} \cdot \nabla T}_{Convection} = \underbrace{k \nabla^2 T}_{Conduction} \tag{4}$$

We now consider several balances by examining the interaction of the hydrodynamic and thermal boundary layers in two regions: the long duct and the short duct . Each is examined below using the method of scale analysis advocated by Bejan [18].

Long Duct Asymptote, $\mathbf{L >> L_t, L >> L_h}$

We begin by considering thermally and hydrodynamically fully developed flow in a non-circular duct of constant cross-section. We write the left hand side of Eq. (4) as:

$$\rho c_p \vec{V} \cdot \nabla T \sim \rho c_p U \frac{(T_w - T_o)}{L} \tag{5}$$

Next considering an enthalpy balance on the duct, we write:

$$\overline{q} P L = \dot{m} c_p (T_w - T_o) \tag{6}$$

Using the above relationship in Eq. (5) we obtain the following result:

$$\rho c_p \vec{V} \cdot \nabla T \sim \frac{\overline{q} P}{A} \tag{7}$$

The energy equation, Eq. (4) for fully developed flow now scales according to

$$\frac{\overline{q} P}{A} \sim k \frac{(T_w - T_o)}{\mathcal{L}^2} \tag{8}$$

where $\mathcal{L}$ represents a characteristic transversal length scale of the duct cross-section. This allows the following relation using the geometry:

$$Nu_{\mathcal{L}} = \frac{\overline{q} \mathcal{L}}{k(T_w - T_o)} \sim \frac{A}{P \mathcal{L}} \tag{9}$$

If the chracteristic length scale is chosen to be $\mathcal{L} = \sqrt{A}$, as done in Part I of this paper, then the following expression is obtained:

$$Nu_{\sqrt{A}} = B \frac{\sqrt{A}}{P} = B_1 \tag{10}$$

In otherwords, the Nusselt number for fully developed flow, scales to a constant related to the geometry of the duct cross-section. This result is valid for both parabolic or uniform velocity distributions.

Short Duct Asymptote, $\mathbf{L << L_t, L >> L_h}$

In the entrance region near the duct inlet two separate problems must be considered. The first assumes that a fully developed hydrodynamic boundary layer exists while the second considers the more general problem where both hydrodynamic and thermal boundary layers develop.

Beginning with the classical Graetz flow problem, where the velocity distribution is assumed to be fully developed and of parabolic distribution, we may write the left hand side of Eq. (4) as:

$$\rho c_p \vec{V} \cdot \nabla T \sim \rho c_p U \frac{\Delta}{\mathcal{L}} \frac{(T_w - T_o)}{L} \tag{11}$$

where Δ is the thermal boundary layer thickness. This assumes that the thermal boundary layer is confined to a region near the duct wall where the velocity distribution is approximately linear, i.e. $|\vec{V}| \sim U\Delta/\mathcal{L}$. The energy equation for Graetz flow now scales according to

$$\rho c_p U \frac{\Delta}{\mathcal{L}} \frac{(T_w - T_o)}{L} \sim k \frac{(T_w - T_o)}{\Delta^2} \tag{12}$$

where the bulk temperature is taken to be equal the inlet temperature, due to a thin thermal boundary layer. Rearranging this expression for Δ yields:

$$\frac{\Delta}{\mathcal{L}} \sim \left(\frac{L}{\mathcal{L} Re_{\mathcal{L}} Pr} \right)^{1/3} \tag{13}$$

Next, considering the heat transfer coefficient which scales according to

$$h(T_w - T_o) \sim \frac{k(T_w - T_o)}{\Delta} \tag{14}$$

we obtain the following expression for the Nusselt number:

$$Nu_{\mathcal{L}} = \frac{h\mathcal{L}}{k} \sim \frac{1}{(L^*)^{1/3}} \tag{15}$$

or

$$Nu_{\mathcal{L}} = \frac{B_2}{(L^*)^{1/3}} \tag{16}$$

where $L^* = L/\mathcal{L} Re_{\mathcal{L}} Pr$, is the dimensionless thermal duct length.

Alternatively, we could have used:

$$\vec{V} \sim \nabla \vec{V} \Delta \sim \frac{\vec{\tau}}{\mu} \Delta \sim f Re_{\mathcal{L}} \frac{U\Delta}{\mathcal{L}} \tag{17}$$

where we have used the result from Part I, i.e. Eq. (9), for the dimensionless average wall shear. This leads to the following well known behaviour:

$$\frac{\Delta}{\mathcal{L}} \sim \left(\frac{L/f Re_{\mathcal{L}}}{\mathcal{L} Re_{\mathcal{L}} Pr} \right)^{1/3} \tag{18}$$

and

$$Nu_{\mathcal{L}} = B_2 \left(\frac{f Re_{\mathcal{L}}}{L^*} \right)^{1/3} \tag{19}$$

Both Eq. (16) and Eq. (19) have the same characteristic and order of magnitude, since $f Re_{\mathcal{L}}$ is constant.

Short Duct Asymptote, $\mathbf{L << L_t, L << L_h}$

Finally, in the combined entrance region, the hydrodynamic boundary layer thickness scales according to the well known expression [18]:

$$\frac{\delta}{L} \sim \frac{1}{(Re_L)^{1/2}} \tag{20}$$

We now consider two distinct regions. These are $\Delta >> \delta$ and $\delta >> \Delta$. When $\Delta >> \delta$, $\vec{V} \sim U$, and the energy equation scales according to

$$\rho c_p U \frac{(T_w - T_o)}{L} \sim k \frac{(T_w - T_o)}{\Delta^2} \tag{21}$$

which gives

$$\frac{\Delta}{L} \sim \frac{1}{(Re_L Pr)^{1/2}} \tag{22}$$

and using Eq. (14), the Nusselt number becomes:

$$Nu_{\mathcal{L}} = \frac{B_3}{(L^*)^{1/2}} \tag{23}$$

When $\Delta << \delta$, $\vec{V} \sim U\Delta/\delta$, and the energy equation scales according to

$$\rho c_p U \frac{\Delta}{\delta} \frac{(T_w - T_o)}{L} \sim k \frac{(T_w - T_o)}{\Delta^2} \tag{24}$$

which gives

$$\frac{\Delta}{L} \sim \frac{1}{Re_L^{1/2} Pr^{1/3}} \tag{25}$$

and using Eq. (14), the Nusselt number becomes:

$$Nu_{\mathcal{L}} = \frac{B_4}{Pr^{1/6} (L^*)^{1/2}} \tag{26}$$

In summary, we have found from scaling analysis the following relationships for the local or average Nusselt number:

$$Nu_{\mathcal{L}} = \begin{cases} B_1 & L >> L_t, L_h \\ B_2 \left(\dfrac{f Re_{\mathcal{L}}}{L^*} \right)^{1/3} & L << L_t, L >> L_h \\ \dfrac{B_3}{(L^*)^{1/2}} & L << L_t, L_h, \; \Delta >> \delta \\ \dfrac{B_4}{Pr^{1/6}(L^*)^{1/2}} & L << L_t, L_h, \; \Delta << \delta \end{cases} \tag{27}$$

This asymptotic behaviour will be examined further and will form the basis for the new model developed for the thermal entrance problem. Finally, an expression relating the approximate magnitude of the thermal entrance length may be obtained by considering an equality between two asymptotic limits given in Eq. (27) for Graetz flow with $L^* = L_t^*$:

$$B_1 = B_2 \left(\frac{f Re_{\mathcal{L}}}{L_t^*} \right)^{1/3} \tag{28}$$

or

$$L_t^* = \left(\frac{B_2}{B_1} \right)^3 f Re_{\mathcal{L}} \tag{29}$$

Later, it will be shown that this approximate scale compares well with exact solutions.

ASYMPTOTIC ANALYSIS

In this section, asymptotic analysis [19] is used to establish expressions for the characteristic long duct and short duct behaviour characterized through scaling analysis. First, the long duct limit is considered yielding a simple relationship for predicting Nusselt number. Additionally, the issue of an appropriate characteristic length scale is addressed. Finally, the short duct limits are considered by re-examining the approximate solution method of Leveque [20] and the laminar boundary layer solutions for isothermal and isoflux plates [1,2,18,19].

Long Duct Asymptote, $\mathbf{L >> L_t, L >> L_h}$

The fully developed flow limit for both hydrodynamic and thermal problems has been addressed by Muzychka [16] and reported in Muzychka and Yovanovich [13,14]. Additional results appear in Yovanovich et al. [15]. These references [13-15] provide models for the classic Graetz problem and the combined entrance problem for forced flow, and for natural convection in vertical isothermal ducts of non-circular cross-section.

In Part I of this paper the authors have shown that when the friction factor-Reynolds number product is based on the square root of cross-sectional area, the large number of data were reduced to a single curve which was a simple function of the aspect ratio of the duct or channel. We also showed that this curve was accurately represented by the first term of the series solution for the rectangular duct cross-section. This important result is given by

$$fRe_{\sqrt{A}} = \frac{12}{\sqrt{\epsilon}(1+\epsilon)\left[1 - \dfrac{192\epsilon}{\pi^5}\tanh\left(\dfrac{\pi}{2\epsilon}\right)\right]} \tag{30}$$

Next, Muzychka [16] and Muzychka and Yovanovich [13] applied the same reasoning to the fully developed Nusselt number in non-circular ducts, leading to the development of a model for the Graetz problem in non-circular ducts. First, it was observed [13,16] that the Nusselt numbers for polygonal ducts collapsed to a single point when $\mathcal{L} = \sqrt{A}$, see Table 1, for both fully developed and slug flows. Next it was observed that when the Nusselt numbers for the rectangular duct and elliptic duct, were normalized with respect to the limiting values for the circular and square ducts, these curves were approximately equal to the normalized friction factor Reynolds number curve, Eq. (30). Finally, a model was been developed which accurately predicts the data for both thermal boundary conditions [16], i.e. (T) and (H). The resulting model which requires Eq. (30) is

$$Nu_{\sqrt{A}} = C_1\left(\frac{fRe_{\sqrt{A}}}{8\sqrt{\pi}\epsilon^{\gamma}}\right) \tag{31}$$

where C_1 is equal to 3.01 for the (UWT) boundary condition and 3.66 for the (UWF) boundary condition. These results are the average value for fully developed flow in a polygonal tube when the characteristic length scale is the square root of cross-sectional area [16]. The parameter γ is chosen based upon the geometry. Values for γ which define the upper and lower bounds in Figs. 1 and 2 are fixed at $\gamma = 1/10$ and $\gamma = -3/10$, respectively. Almost all of the available data are predicted within ± 10 percent by Eq.(31), with few exceptions.

Figures 1-3 compare the data for many duct shapes obtained from Shah and London [1] for both singly and doubly connected ducts. When the results are based upon $\mathcal{L} = \sqrt{A}$, and an appropriate aspect ratio defined, two distinct bounds are formed in Figs. 1 and 2 for the Nusselt number. The lower bound consists of all duct shapes which have re-entrant corners, i.e. angles less than 90 degrees, while the upper bound consists of all ducts with rounded corners and/or right angled corners.

Figure 3 shows the Nusselt number for the (H) condition for polygonal annular ducts and the circular annular duct. The predicted values are determined from Eq. (31) using the equivalent definition of aspect ratio given in Part I for the circular annulus. For the polygonal annular ducts, the equivalent $r^* = \sqrt{A_i/A_o}$, where A_i and A_o are the cross-sectional areas of the inner and outer ducts respectively.

Table 1
Nusselt Numbers for Slug and Fully Developed Flow for Regular Polygons

		Isoflux (H)		Isothermal (T)	
	Geometry	**FDF**	**Slug**	**FDF**	**Slug**
Nu_{D_h}	Triangle	3.11	-	2.47	-
	Square	3.61	7.08	2.98	4.93
	Hexagon	4.00	7.53	3.35	5.38
	Octagon	4.21	7.69	3.47	5.53
	Circular	4.36	7.96	3.66	5.77
$Nu_{\sqrt{A}}$	Triangle	3.51	-	2.79	-
	Square	3.61	7.08	2.98	4.93
	Hexagon	3.74	7.01	3.12	5.01
	Octagon	3.83	7.00	3.16	5.03
	Circular	3.86	7.06	3.24	5.11

Short Duct Asymptote, $\mathbf{L << L_t, L >> L_h}$

If the velocity distribution is fully developed and the temperature distribution is allowed to develop, the classic Graetz problem results. In the thermal entrance region, the results are weak functions of the shape and geometry of the duct. This behaviour is characterized by the following approximate analytical expression attributed to Leveque, see [20]:

$$Nu \propto \left(\frac{C^*}{L^*}\right)^{1/3} \tag{32}$$

where C^* is the dimensionless mean velocity gradient at the duct wall and L^* is the dimensionless axial location. Thus, if C^* is a weak function of shape, then Nu will be a weaker function of shape due to the one third power.

In the thermal entrance region of non-circular ducts the thermal boundary layer is thin and is assumed to be developing in a region where the velocity gradient is linear. For very small distances from the duct inlet, the effect of curvature on the boundary layer development is negligible. Thus, the analysis considers the duct boundary as a flat plate. The governing equation for this situation is given by

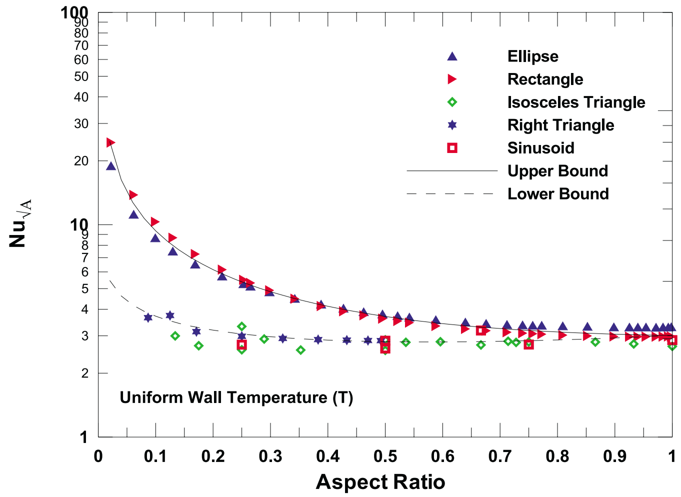

Fig. 1 - Fully Developed Flow $\mathrm{Nu_T}$, data from Ref.[1].

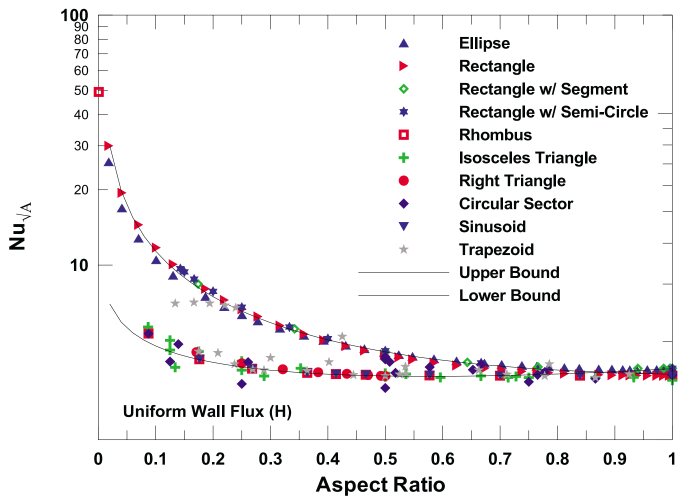

Fig. 2 - Fully Developed Flow $\mathrm{Nu_H}$, data from Ref.[1].

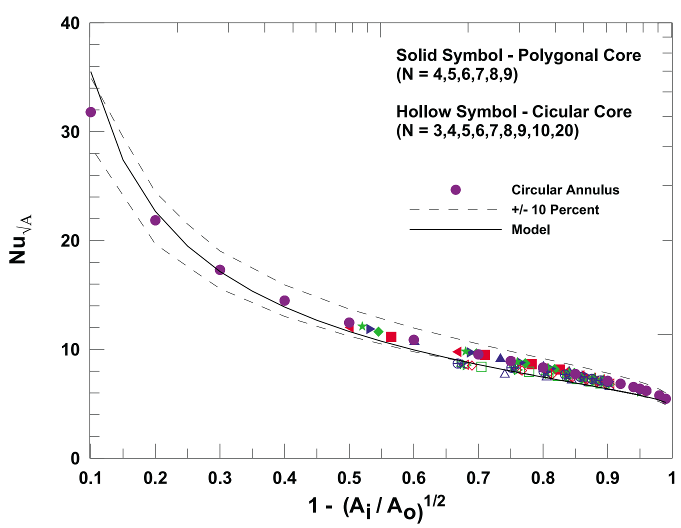

Fig. 3 - Fully Developed Flow $\mathrm{Nu_H}$ in Annular Ducts, data from Ref. [1].

$$Cy\frac{\partial T}{\partial z} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{33}$$

where the constant C represents the mean velocity gradient at the duct wall. For non-circular ducts, this constant is defined as:

$$C = \overline{\left.\frac{\partial u}{\partial n}\right|_w} = \frac{1}{P}\oint \left.\frac{\partial u}{\partial n}\right|_w ds \tag{34}$$

For hydrodynamically fully developed flow, the constant C is related to the friction factor-Reynolds number product

$$\frac{fRe_{\mathcal{L}}}{2} = \overline{\left.\frac{\partial u}{\partial n}\right|_w}\frac{\mathcal{L}}{U} = C^* \tag{35}$$

where $\mathcal{L}$ is an arbitrary length scale.

Defining the following parameters:

$$Y = \frac{y}{\mathcal{L}} \qquad Z = \frac{z/\mathcal{L}}{Re_{\mathcal{L}}Pr} \qquad \theta = T - T_o \qquad Re_{\mathcal{L}} = \frac{U\mathcal{L}}{\nu}$$

leads to the governing equation

$$C^*Y\frac{\partial \theta}{\partial Z} = \frac{\partial^2 \theta}{\partial Y^2} \tag{36}$$

The governing equation can now be transformed into an ordinary differential equation for each wall condition using a similarity variable [20]

$$\eta = \frac{Y}{(9Z/C^*)^{1/3}} \tag{37}$$

Both the UWT and UWF conditions are examined. Solutions to the Leveque problem are discussed in Bird et al. [20]. The solution for local Nusselt number with the UWT condition yields:

$$Nu_{\mathcal{L}} = \frac{3\sqrt{3}\,\Gamma(2/3)}{2\pi}\left(\frac{fRe_{\mathcal{L}}}{18Z}\right)^{1/3} = 0.4273\left(\frac{fRe_{\mathcal{L}}}{Z}\right)^{1/3} \tag{38}$$

while the solution for the local Nusselt number for the (UWF) condition yields:

$$Nu_{\mathcal{L}} = \Gamma(2/3)\left(\frac{fRe_{\mathcal{L}}}{18Z}\right)^{1/3} = 0.5167\left(\frac{fRe_{\mathcal{L}}}{Z}\right)^{1/3} \tag{39}$$

The average Nusselt number for both cases can be obtained by integrating Eqs. (38, 39):

$$\overline{Nu}_{\mathcal{L}} = \frac{3}{2}Nu_{\mathcal{L}} \tag{40}$$

The solution for each wall condition can be compactly written with $Z = L^*$ as:

$$Nu_{\mathcal{L}} = C_2C_3\left(\frac{fRe_{\mathcal{L}}}{L^*}\right)^{1/3} \tag{41}$$

where the value of C_2 is 1 for local conditions and 3/2 for average conditions, and C_3 takes a value of 0.427 for UWT and 0.517 for UWF.

The Leveque approximation is valid where the thermal boundary layer is thin, in the region near the wall where the velocity profile is linear. This corresponds to flow of large Prandtl number fluids, which give rise to very short hydrodynamic entrance lengths, i.e. hydrodynamically fully developed flow. The weak effect of duct geometry in the entrance region is due to the presence of the friction factor-Reynolds number product, fRe, in the above expression, which is representative of the average velocity gradient at the duct wall. The typical range of fRe is $12 < fRe_{D_h} < 24$, Shah and London [1]. This results in $2.29 < (fRe_{D_h})^{1/3} < 2.88$, which illustrates the weak dependency of the thermal entrance region on shape and aspect ratio. Further reductions are achieved for similar shaped ducts by using the length scale $\mathcal{L} = \sqrt{A}$.

Short Duct Asymptote, $\mathbf{L << L_t, L << L_h}$

Finally, if both hydrodynamic and thermal boundary layers develop simultaneously, the results are strong functions of the fluid Prandtl number. In the combined entrance region the behaviour for very small values of L^* may be adequately modelled by treating the duct wall as a flat plate. The characteristics of this region are:

$$\frac{Nu_z}{\sqrt{Re_z}} = \begin{cases} 0.564 Pr^{1/2} & Pr \to 0 \\ 0.339 Pr^{1/3} & Pr \to \infty \end{cases} \tag{42}$$

for the UWT condition [4], and

$$\frac{Nu_z}{\sqrt{Re_z}} = \begin{cases} 0.886 Pr^{1/2} & Pr \to 0 \\ 0.464 Pr^{1/3} & Pr \to \infty \end{cases} \tag{43}$$

for the UWF condition [3].

Composite models for each wall condition were developed by Churchill and Ozoe [3,4] using the asymptotic correlation method of Churchill and Usagi [17]. The results can be developed in terms of the $Pr \to 0$ behaviour or the $Pr \to \infty$ behaviour. For internal flow problems, the appropriate form is chosen to be in terms of the $Pr \to 0$ characteristic which introduces the Peclet number $Pe = RePr$:

$$\frac{Nu_z}{(Re_z Pr)^{1/2}} = \frac{C_o}{\left[1 + \left(\dfrac{C_o Pr^{1/6}}{C_\infty}\right)^n\right]^{1/n}} = f(Pr) \tag{44}$$

where C_o and C_∞ represent the coefficients of the right hand side of Eqs. (42, 43). The correlation parameter n is found by solving Eq. (44) at an intermediate value of Pr where the exact solution is known, i.e. $Pr = 1$. This leads to $n = 4.537$ for the UWT condition and $n = 4.598$ for the UWF condition. For simplicity, $n = 9/2$ is chosen for both cases.

The average Nusselt number for both cases is now obtained by integrating Eq. (44):

$$\overline{Nu}_{\mathcal{L}} = 2 Nu_{\mathcal{L}} \tag{45}$$

After introducing L^*, the solution for each wall condition can be compactly written as:

$$Nu_{\mathcal{L}} = C_4 \frac{f(Pr)}{\sqrt{L^*}} \tag{46}$$

where the value of $C_4 = 1$ for local conditions and $C_4 = 2$ for average conditions, and $f(Pr)$ is defined as:

$$f(Pr) = \frac{0.564}{\left[1 + \left(1.664 Pr^{1/6}\right)^{9/2}\right]^{2/9}} \tag{47}$$

for the UWT condition, and

$$f(Pr) = \frac{0.886}{\left[1 + \left(1.909 Pr^{1/6}\right)^{9/2}\right]^{2/9}} \tag{48}$$

for the UWF condition. The preceding results are valid only for small values of L^*.

MODEL DEVELOPMENT AND COMPARISONS

A model which is valid over the entire range of dimensionless duct lengths for $Pr \to \infty$, was developed by Muzychka and Yovanovich [13] by combining Eq. (31) with Eq. (41) using the Churchill and Usagi [17] asymptotic correlation method. The form of the proposed model for an arbitrary characteristic length scale is:

$$Nu(z^*) = \left(\left\{C_2 C_3 \left(\frac{fRe}{z^*}\right)^{\frac{1}{3}}\right\}^n + (Nu_{fd})^n\right)^{1/n} \tag{49}$$

Now using the result for the fully developed friction factor, Eq. (32), and the result for the fully developed flow Nusselt number, Eq. (33), with $n \approx 5$ a new model [13] was proposed having the form:

$$Nu_{\sqrt{A}}(L^*) = \left[\left\{C_2 C_3 \left(\frac{fRe_{\sqrt{A}}}{L^*}\right)^{\frac{1}{3}}\right\}^5 + \left\{C_1 \left(\frac{fRe_{\sqrt{A}}}{8\sqrt{\pi}\epsilon^\gamma}\right)\right\}^5\right]^{\frac{1}{5}} \tag{50}$$

where the constants C_1, C_2, C_3 and γ are given in Table 2. These constants define the various cases for local or average Nusselt number and isothermal or isoflux boundary conditions for the Graetz problem. The constant C_2 was modified from that found by the Leveque approximation to provide better agreement with the data.

A model for the combined entrance region is now developed by combining the solution for a flat plate with the model for the Graetz flow problem developed earlier. The proposed model takes the form:

$$Nu_{\sqrt{A}}(L^*, Pr) = \left[\left(\left\{C_2 C_3 \left(\frac{fRe_{\sqrt{A}}}{L^*}\right)^{\frac{1}{3}}\right\}^5 + \left\{C_1 \left(\frac{fRe_{\sqrt{A}}}{8\sqrt{\pi}\epsilon^\gamma}\right)\right\}^5\right)^{m/5} + \left\{C_4 \frac{f(Pr)}{\sqrt{L^*}}\right\}^m\right]^{1/m} \tag{51}$$

which is similar to that proposed by Churchill and Ozoe [3,4] for the circular duct. This model is a composite solution of the three asymptotic solutions just presented.

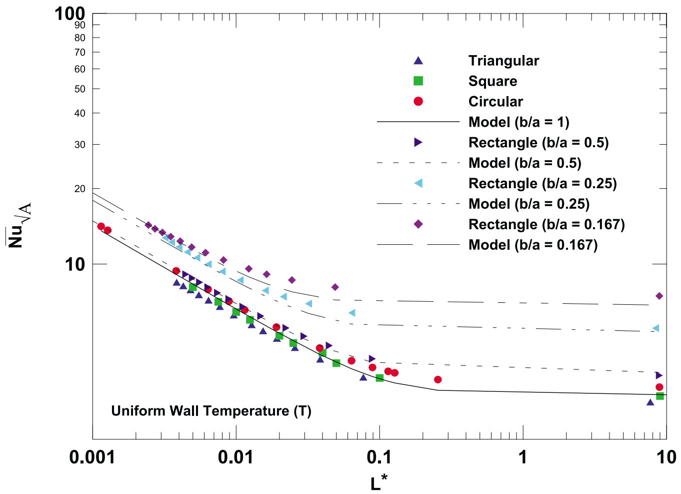

Fig. 4 - Thermally Developing Flow $Nu_{T,m}$, data from Ref.[1].

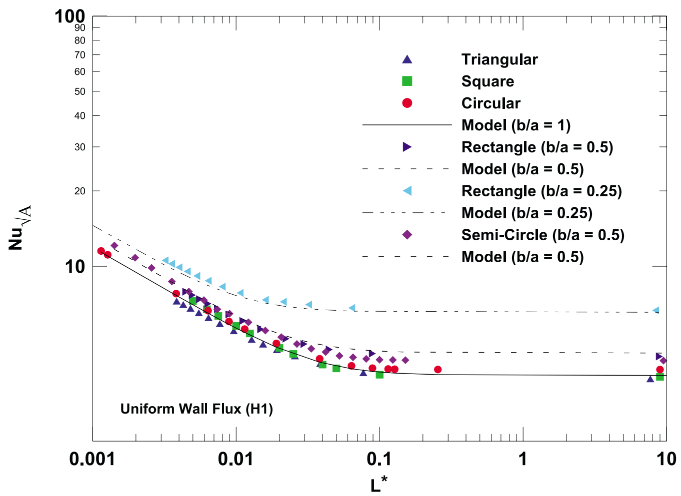

Fig. 5 - Thermally Developing Flow $Nu_{H,z}$, data from Ref.[1].

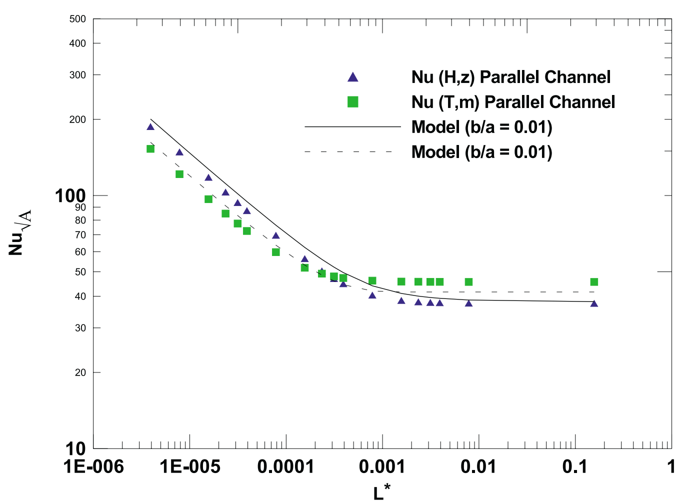

Fig. 6 - Thermally Developing Flow in Parallel Plate Channel, data from Ref.[1].

The parameter m was determined to lie in the range $2 < m < 7$, for all data examined. Values for the blending parameter were found to be weak functions of the duct aspect ratio and whether a local or average Nusselt number was considered. However, the blending parameter was found to be most dependent upon the fluid Prandtl number.

A simple linear approximation was determined to provide better accuracy than choosing a single value for all duct shapes. Due to the variation in geometries and data, higher order approximations offered no additional advantage. Therefore, the linear approximation which predicts the blending parameter within 30 percent was found to be satisfactory. Variations in the blending parameter of this order will lead to small errors in the model predictions, whereas variations on the order of 100 percent or more, i.e. choosing a fixed value, produce significantly larger errors. The resulting fit for the correlation parameter m is:

$$m = 2.27 + 1.65 Pr^{1/3} \qquad (52)$$

The above model is valid for $0.1 < Pr < \infty$ which is typical for most low Reynolds number flow heat exchanger applications.

Comparisons with the available data from Ref. [1] are provided in Tabular form in [13,14,16] and Figs. 4-9. Good agreement is obtained with the data for the circular duct and parallel plate channel. Note that comparison of the model for the parallel plate channel was obtained by considering a rectangular duct having an aspect ratio of $\epsilon = 0.01$. This represents a reasonable approximation for this system. The data are also compared with the models of Churchill and Ozoe [3,4] and Stephan [5,6] for the circular duct and parallel plate channel in Figs. 7 and 8.

The numerical data for the UWT circular duct fall short of the model predictions at low Pr numbers. However, all of the models are in excellent agreement with the integral formulation of Kreith [22]. Good agreement is also obtained for the case of the square duct for all Prandtl numbers. Comparisons of the model with data for the rectangular duct at various aspect ratios and the equilateral triangular duct show that larger discrepancies arise. These data neglect the effects of transverse velocities in both the momentum and energy equations. Additional graphical comparison can be found in [14].

The accuracy for each case may be improved considerably by using the optimal value of the parameter m. However, this introduces an additional parameter into the model which is deemed unnecessary for purposes of heat exchanger design. The proposed model predicts most of the available data for the combined entry problem to within ± 15 percent and most of the data for the data for Graetz flow within $\pm$ 12 percent [13,14,16]. These models can also be used to predict the heat transfer characteristics for other non-circular ducts for which there are presently no data.

The present models also agree well with the published models of Churchill and Ozoe [3,4] and the models of Stephan [5,6]. It is evident from Figs. 4-9 that the present models provide acceptable accuracy for design calculations.

Thermal Entrance Lengths

An equation for predicting the thermal entrance length is now obtained from Eq. (29) with B_1 corresponding to Eq.

(31) and $B_2 = C_2 C_3$. This gives the following relationship:

$$L_t^* = 19.80 \left(\frac{C_2 C_3}{C_1} \right)^3 \epsilon^{1+3\gamma} (1+\epsilon)^2 \left[1 - \frac{192}{\pi^5} \tanh \left(\frac{\pi}{2\epsilon} \right) \right]^2 \quad (53)$$

Using Eq. (53) with the appropriate values of C_1, C_2, C_3, γ yields, $L_t^* = 0.0358$ and $L_t^* = 0.0366$ for the circular duct for (T) and (H) respectively. When rescaled to be based upon the hydraulic diameter, we obtain $L_t^* = 0.0281$ and $L_t^* = 0.0287$ for the circular duct for (T) and (H), which compare with the values from [1], $L_t^* = 0.0335$ and $L_t^* = 0.0430$ for (T) and (H), respectively.

SUMMARY AND CONCLUSIONS

A general model for predicting the heat transfer coefficient in the combined entry region of non-circular ducts was developed. This model is valid for $0.1 < Pr < \infty$, $0 < L^* < \infty$, both uniform wall temperature (UWT) and uniform wall flux (UWF) conditions, and for local and mean Nusselt numbers. Model predictions agree with numerical data to within ± 15 percent or better for most non-circular ducts and channels. The model was developed by combining the asymptotic results of laminar boundary layer flow and Graetz flow for the thermal entrance region. In addition, by means of a novel characteristic length, the square root of cross-sectional area, results for many non-circular ducts of similar aspect ratio collapse onto a single curve.

The present study took advantage of scale analysis, asymptotic analysis, and the selection of a more appropriate characteristic length scale to develop a simple model. This model only requires two parameters, the aspect ratio of the duct and the dimensionless duct length. Whereas the models of Yilmaz and Cihan [11,12] for Graetz flow consist of several equations. The present model predicts most of the developing flow data within ± 12 percent or better.

Finally these models may also be used to predict results for ducts for which no solutions or tabulated data exist.

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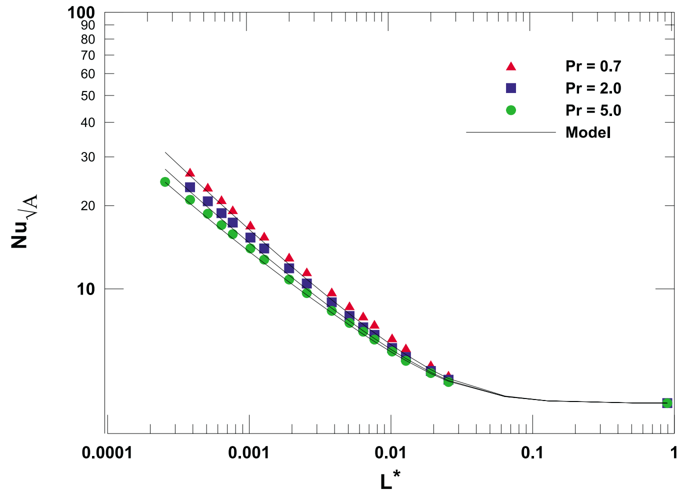

Fig. 7 - Simultaneously Developing Flow in a Circular Tube $Nu_{H,z}$, data from Ref. [1] .

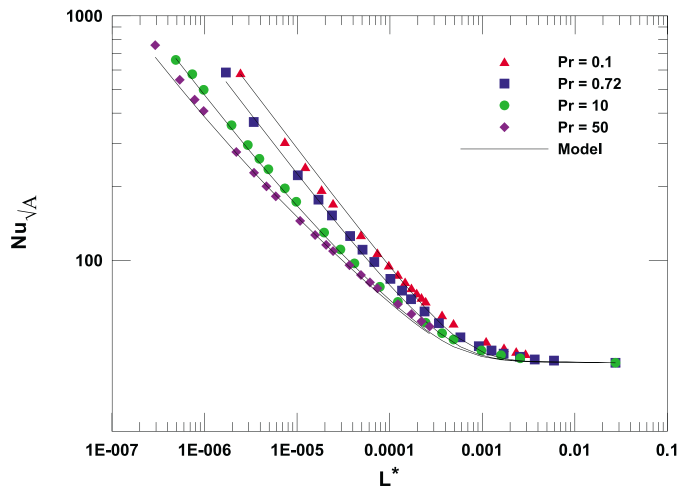

Fig. 8 - Simultaneously Developing Flow in Channel $Nu_{T,m}$, data from Ref. [1].

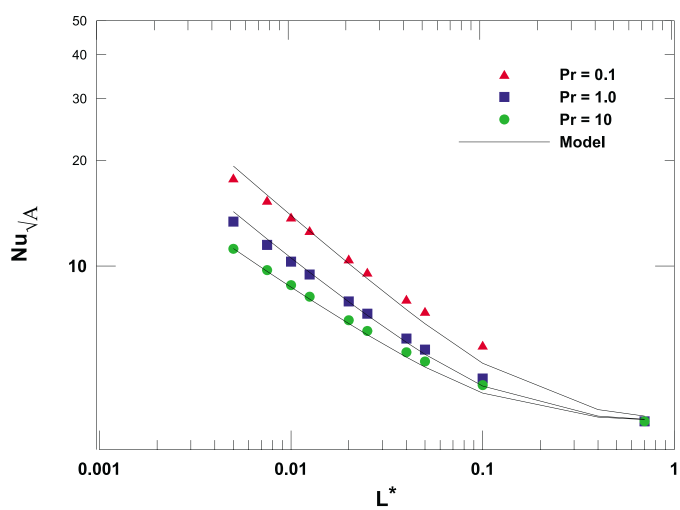

Fig. 9 - Simultaneously Developing Flow in Square Duct $Nu_{H,m}$, data from Ref. [1].

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Table 2
Coefficients for General Model

Boundary Condition		
UWT (T)	$C_1 = 3.01,\ \ C_3 = 0.409$	$f(Pr) = \dfrac{0.564}{\left[1+(1.664 Pr^{1/6})^{9/2}\right]^{2/9}}$
UWF (H)	$C_1 = 3.66,\ \ C_3 = 0.501$	$f(Pr) = \dfrac{0.886}{\left[1+(1.909 Pr^{1/6})^{9/2}\right]^{2/9}}$
Nusselt Number Type		
Local	$C_2 = 1$	$C_4 = 1$
Average	$C_2 = 3/2$	$C_4 = 2$
Shape Parameter		
Upper Bound	$\gamma = 1/10$	
Lower Bound	$\gamma = -3/10$	