
Week 3

Lecture 1

• Vibrating String and Membranes (Rectangular and Circular). The 1-D wave equation for the string in cartesian coordinates is

$$u_{xx} = \frac{1}{c^2} u_{tt}, \qquad t > 0, \qquad 0 < x < L$$

It can be modified to include vibrations of rectangular membranes where the displacement from equilibrium is $u(x, y, t)$ and circular membranes where for axisymmetric vibrations $u(r, t)$.

• Rectangular Membrane
The 2-D wave equation for rectangular membranes is

$$u_{xx} + u_{yy} = \frac{1}{c^2} u_{tt}, \quad t > 0, \quad -L < x < L, \quad -W < y < W$$

where $c^2 = T/\rho$ and now ρ is the mass density per unit area of the membrane.

• Circular Membrane
The 1-D axisymmetric wave equation for a circular membrane of radius a is

$$u_{rr} + \frac{1}{r} u_r = \frac{1}{c^2} u_{tt}, \qquad t > 0, \qquad 0 < r < a$$

where $c^2 = T/\rho$ and now ρ is the mass density per unit area of the membrane.

• Laplace Equation in Cartesian Coordinates
Derivation of continuity equation (conservation of mass) for two-dimensional, steady flow of an incompressible fluid. The procedure is presented below.

• Select a differential control volume (CV): $dV = dxdy1$ of unit length into the paper.
• Velocity components are $u(x, y)$ and $v(x, y)$ along the $x-$ and $y-$coordinates respectively.

• Mass flow rates into the CV are $\dot{m}_x = \rho u dA_x$ and $\dot{m}_y = \rho v dA_y$ where $dA_x = 1dy$ and $dA_y = 1dx$.
• Mass flow rate out of the CV through the opposite faces located at $x + dx$ and $y + dy$ are (taking the first two terms of the Taylor series expansions of $\dot{m}_x$ and $\dot{m}_y$):

$$\dot{m}_x + \frac{\partial \dot{m}_x}{\partial x} dx \qquad \text{and} \qquad \dot{m}_y + \frac{\partial \dot{m}_y}{\partial y} dy$$

• Conservation of mass principle for steady flow requires that the mass flow rate into the CV must equal the mass flow rate out of the CV. Thus

$$\dot{m}_x + \dot{m}_y = \dot{m}_x + \frac{\partial \dot{m}_x}{\partial x} dx + \dot{m}_y + \frac{\partial \dot{m}_y}{\partial y} dy$$

Cancelling terms and substituting for the mass flow rates and the flow areas gives

$$\left[\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} \right] dx dy 1 = 0$$

Dividing by the differential volume gives the 2-D partial differential equation for steady flow in the absence of mass sources. It is called the continuity equation:

$$\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0$$

The above equation is applicable for variable mass density, i.e. $\rho(x, y)$. It can be expanded and written in the following form:

$$\rho \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right] + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} = 0$$

• Vector Form of Continuity Equation

$$\rho \nabla \cdot \vec{V} + \left(\vec{V} \cdot \nabla \right) \rho = 0$$

where the velocity vector is $\vec{V} = \vec{i} u + \vec{j} v$. Consult your fluids text for details.

• Incompressible Fluid
If $\rho =$ constant, then the continuity equation becomes

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

• Velocity Potential.
Here we introduce the velocity potential $\phi(x, y)$ such that the velocity components are given by the relations:

$$u = \frac{\partial \phi}{\partial x} \qquad \text{and} \qquad v = \frac{\partial \phi}{\partial y}$$

Consult your fluids text for details.

• Laplace Equation
Substitution into the continuity equation gives the 2-D Laplace equation in cartesian coordinates:

$$\nabla^2 \phi = \nabla \cdot \nabla \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

which describes the 2-D steady flow of an incompressible fluid. If the velocity vector is three-dimensional, i.e. $\vec{V} = \vec{i}u(x, y, z) + \vec{j}v(x, y, z) + \vec{k}w(x, y, z)$, then the continuity equation can be expressed as

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

and

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

• The Laplace equation also appears in many other physical problems such as electricity and magnetism, steady heat conduction and steady mass transfer, gravitational potential, etc. Functions which satisfy (are solutions of) the Laplace equation are called harmonic functions.

See Problem Set 1 for examples of two- and three-dimensional harmonics which satisfy the two- and three-dimensional forms of the Laplace equation.

Lecture 2

• Derivation of Heat Equation Diffusion Equation (Heat Equation) with Distributed Heat Sources.

• Select a three-dimensional control volume in cartesian coordinates: $dV = dxdydz$.
• Use Fourier's Law of Conduction:

$$\vec{q} = -k\nabla T \quad \text{or} \quad \vec{i}\, q_x + \vec{j}\, q_y + \vec{k}\, q_z = -k\left[\vec{i}\,\frac{\partial T}{\partial x} + \vec{j}\,\frac{\partial T}{\partial y} + \vec{k}\,\frac{\partial T}{\partial z}\right]$$

The heat flux components are

$$q_x = -k\,\frac{\partial T}{\partial x}, \quad q_y = -k\,\frac{\partial T}{\partial y}, \quad q_z = -k\,\frac{\partial T}{\partial z}$$

and the temperature is time dependent, i.e. $T = T(x, y, z, t)$. The thermal conductivity k depends on the temperature, i.e. $k = k(T)$. The three heat flow rates into the CV along the three coordinate directions are:

$$Q_x = q_x\,dA_x = -k\frac{\partial T}{\partial x}dydz, \quad Q_y = q_y\,dA_y = -k\frac{\partial T}{\partial y}dxdz, \quad Q_z = q_z\,dA_z = -k\frac{\partial T}{\partial z}dxdy$$

The three heat flow rates out of the CV through the opposite faces located at $x + dx, y + dy, z + dz$ are respectively

$$Q_x + \frac{\partial Q_x}{\partial x}dx = Q_x - \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) dxdydz$$

$$Q_y + \frac{\partial Q_y}{\partial y}dy = Q_y - \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) dxdydz$$

$$Q_z + \frac{\partial Q_z}{\partial z}dz = Q_z - \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) dxdydz$$

The *net* heat flow rate into the CV is equal to the difference between the heat flow rates into and out of the CV. Therefore the *net* heat flow rate into the CV through the six faces of the CV is given by

$$\left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right] dxdydz$$

If distributed volumetric heat sources are present, i.e. $\mathcal{P} > 0$, then there is heat generation in the CV, and

$$Q_{\text{gen}} = \mathcal{P}dxdydz$$

During transient heating, there is internal energy storage in the CV. Thus

$$Q_{\text{storage}} = \rho c_p \frac{\partial T}{\partial t}dxdydz$$

where ρ is the mass density and c_p is the specific heat capacity of the material.

• Conservation of Energy Principle.
Conservation of energy principle can be stated as

Net Conduction Rate into CV+Heat Generation Rate in CV = Energy Storage in CV

This leads to the three-dimensional form of the transient Diffusion Equation with distributed volumetric heat sources and temperature dependent thermal conductivity:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\mathcal{P}=\rho c_p\frac{\partial T}{\partial t}, \quad t>0$$

after dividing by the common differental volume $dV = dxdydz$. There are three conduction terms, one source term and one storage term in the general Diffusion Equation. If the thermal conductivity is constant, i.e. k = constant, we can factor k, and then divide through by k to get the alternative form of the Diffusion Equation:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\mathcal{P}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}, \quad t>0$$

where the thermophysical parameter is $\alpha = k/(\rho c_p) > 0$.

- Vector Form of Transient Diffusion Equation With Distributed Heat Sources.

$$\nabla^2 T+\frac{\mathcal{P}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}, \quad t>0$$

- Vector Form of Transient Diffusion Equation Without Distributed Heat Sources.

$$\nabla^2 T=\frac{1}{\alpha}\frac{\partial T}{\partial t}, \quad t>0$$

- Vector Form of Steady Diffusion Equation With Distributed Heat Sources.

$$\nabla^2 T=-\frac{\mathcal{P}}{k}$$

This PDE is called Poisson's equation and $T(x, y, z)$. It is an elliptic type.

- Vector Form of Steady Diffusion Equation Without Distributed Heat Sources.

$$\nabla^2 T=0$$

This PDE is called Laplace's equation and $T(x, y, z)$. It is an elliptic type.

- Solution of these equations will be obtained.

Lecture 3

• Nondimensionalization of the one-dimensional diffusion equation $T_{xx} = \frac{1}{\alpha}T_t$ defined on the finite interval: $0 \le x \le L$. The initial condition $T(x,0) = T_i$ and the two boundary conditions: $T(0,t) = T_0, T(L,t) = T_i < T_0$ are non-homogeneous. The dependent variable (temperature) depends on six independent parameters: $T = T(x, L, t, \alpha, T_i, T_0)$. The dimensionless position: $\xi = x/L$, dimensionless time: $\tau = \alpha t/L^2$, and the dimensionless temperature: $\phi = (T(x,t) - T_i)/(T_0 - T_i)$ lead to the dimensionless form of the PDE, BCs and IC.

$$\phi_{\xi\xi} = \phi_\tau, \qquad \tau > 0, \qquad 0 < \xi < 1$$

and

$$\phi(\xi, 0) = 0, \qquad \phi(0,\tau) = 1, \qquad \phi(1,\tau) = 0$$

The PDE is homogeneous, the IC is homogeneous, and one of the two BCs is nonhomogeneous. The dimensionless temperature depends on two independent dimensionless parameters: $\phi(\xi, \tau)$. See the ME 303 Web site for details of the method.

• Fourier Cosine and Sine Series.

• See the summary in Spiegel's Handbook of Mathematics, pages 131-135, and Spiegel's text, pages 382-383, and the examples on pages 383-395. See the material and the Maple worksheets on the ME 303 Web site.

• Fourier cosine and sine series are very important in the solutions of the one-dimensional wave and diffusion equations and the two-dimensional Laplace equation formulated in cartesian coordinates.
