Week 12

Lecture 1

- Solution of ODE

$$\frac{d\theta}{dt} + m\,\theta = n, \quad t > 0, \quad \text{IC} \quad \theta(0) = \theta_i$$

where $\theta(t) = T(t) - T_\infty$, and the constants are:

$$m = \frac{hA}{\rho c_p V} \quad \text{and} \quad n = \frac{q_i A}{\rho c_p V}$$

where A = surface area, V = volume, q_i = incident heat flux. Units are: $h\ [W/(m^2 \cdot K)]$, $q_i\ [W/m^2]$, $m\ [1/s]$, and $n\ [K/s]$

- Solution of Problem (SVM). Separate the variables and integrate.

$$\frac{d\theta}{\theta - \dfrac{n}{m}} = -m\,dt$$

Write as

$$\frac{d\left(\theta - \dfrac{n}{m}\right)}{\theta - \dfrac{n}{m}} = -m\,dt$$

Integrate to get

$$\ln\left(\theta(t) - \frac{n}{m}\right) = -m\,t + \ln C_1$$

Apply IC to find the constant of integration:

$$C_1 = \ln\left(\theta_i - \frac{n}{m}\right)$$

Substitute to get the solution

$$\theta(t) = \frac{n}{m} + \left(\theta_i - \frac{n}{m}\right) e^{-mt}, \quad t > 0$$

The steady-state solution occurs when $t = \infty$:

$$\theta(\infty) = \frac{n}{m}$$

- Laplace Transform Method

$$\mathcal{L}\left\{\frac{d\theta}{dt}\right\} + \mathcal{L}\left\{m\,\theta(t)\right\} = \mathcal{L}\left\{n\right\}$$

which can be written as

$$s\bar{\theta}(s) - \theta(0) + m\,\bar{\theta}(s) = \frac{n}{s}$$

Substitute the IC and solve for $\bar{\theta}(s)$.

$$\bar{\theta}(s) = \frac{n}{s(s+m)} + \frac{\theta_i}{s+m}$$

The solution is obtained by taking the inverse Laplace transform:

$$\theta(t) = \mathcal{L}^{-1}\left\{\bar{\theta}(s)\right\} = \mathcal{L}^{-1}\left\{\frac{n}{s(s+m)}\right\} + \mathcal{L}^{-1}\left\{\frac{\theta_i}{s+m}\right\}$$

To obtain the inverse Laplace transform of the first term on the right hand side, we must use partial fractions. Write

$$\frac{n}{s(s+m)} = n\left(\frac{A}{s} + \frac{B}{s+m}\right) = n\left(\frac{A(s+m) + Bs}{s(s+m)}\right)$$

Now equate

$$s(A+B) = 0 \quad \text{and} \quad Am = 1$$

Therefore the constants are

$$A = \frac{1}{m} \quad \text{and} \quad B = -A = -\frac{1}{m}$$

Now we can write the first term as

$$\frac{n}{s(s+m)} = \frac{n}{ms} - \frac{n}{m(s+m)}$$

Taking the inverse Laplace transform gives

$$\theta(t) = \mathcal{L}^{-1}\left\{\frac{n}{ms} - \frac{n}{m(s+m)} + \frac{\theta_i}{s+m}\right\}$$

and

$$\theta(t) = \left(\frac{n}{m}\right)\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \left(\theta_i - \frac{n}{m}\right)\mathcal{L}^{-1}\left\{\frac{1}{s+m}\right\}$$

which gives for the solution

$$\theta(t) = \frac{n}{m} + \left(\theta_i - \frac{n}{m}\right)e^{-mt}, \quad t > 0$$

See the ME 303 Web site for Maple worksheets which deal with this and related problems.
The tutorial will cover the application of the Laplace transform method to a second order differential equation from dynamics.

- Laplace Transform Method Application
- System of ODEs

$$x'(t) + 2y'(t) - 2y(t) = t$$

and

$$x(t) + y'(t) - y(t) = 1$$

with ICs:

$$x(0) = 0, \quad \text{and} \quad y(0) = 0$$

Apply Laplace transform method to ODES and substitute the ICs to get two algebraic equations:

$$1) \quad sX(s) + 2sY(s) - 2Y(s) = \frac{1}{s^2}$$

and

$$2) \quad X(s) + sY(s) - Y(s) = \frac{1}{s}$$

¿From 2) we find

$$X(s) = \frac{1}{s} - sY(s) + Y(s)$$

Substitute into 1) to get

$$s\left[\frac{1}{s} - sY(s) + Y(s) + 2sY(s) - 2Y(s)\right] = \frac{1}{s^2}$$

or

$$(s^2 - 3s + 2)Y(s) = 1 - \frac{1}{s^2}$$

Solving for $Y(s)$:

$$Y(s) = \frac{s+1}{s^2(s-2)} = -\frac{3}{4} \cdot \frac{1}{s} - \frac{1}{2} \cdot \frac{1}{s^2} + \frac{3}{4} \cdot \frac{1}{s-2}$$

Take inverse Laplace transform to get

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{3}{4} - \frac{t}{2} + \frac{3}{4}e^{2t}$$

Now we can solve for $X(s)$, then take the inverse Laplace transform to obtain $x(t)$. A simpler way is to substitute the result for $y(t)$ into the second ODE to find

$$x(t) = \frac{3}{4} - \frac{3}{4}e^{2t} - \frac{t}{2}$$

Lecture 2

• Makeup Lecture

• See ME 303 Web site for material on Laplace Transform Method applied to PDEs.

• Laplace Transform of Partial Derivatives: Space and Time Derivatives.

$$\mathcal{L}\left\{\frac{\partial u}{\partial x}\right\} = \frac{d}{dx}\mathcal{L}\{u\} \equiv \frac{d\overline{u}}{dx}(x,s)$$

and

$$\mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_0^\infty e^{-st}\,\frac{\partial^2 u}{\partial x^2}\,dt = \frac{d^2}{dx^2}\int_0^\infty e^{-st}\,u\,dt = \frac{d^2\overline{u}}{dx^2}\,(x,s)$$

and

$$\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = s\overline{u}(x,s) - u(x,0)$$

and

$$\mathcal{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2\overline{u}(x,s) - su(x,0) - \frac{\partial u}{\partial t}(x,0)$$

• See material on the application of Laplace Transform Method to the solution of the 1D heat equation in a half-space $x > 0$ for the Dirichlet boundary condition.

Summary of Laplace Transform Solution of Diffusion Equation, BCs and IC

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s-Domain	t-Domain
$\dfrac{d^2\overline{u}}{dx^2} - \dfrac{s}{\alpha}\overline{u} = 0$	$\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial u}{\partial t}$
$\overline{u}(x,s) = u_0\,\dfrac{1}{s}e^{-\sqrt{s/\alpha}\,x}$	$u(x,t) = u_0\; erfc\left(\dfrac{x}{2\sqrt{\alpha t}}\right)$
At $x = 0$,	
$\overline{u}(0,s) = \dfrac{u_0}{s}$	$u(0,t) = u_0$
As $x \to \infty$,	
$\overline{u}(\infty,s) = 0$	$u(\infty,t) = 0$

Lecture 3

- ME 303 Web site has short Laplace Transform Table with entries not found in Schaum's Outline. They are applicable for solutions of diffusion equation in half-space with boundary conditions: i) Dirichlet, ii) Neumann, and iii) Robin.

- Laplace transform solution of diffusion equation in finite region: $0 \le x \le a$.

$$\text{PDE}: \qquad u_{xx} = \frac{1}{\alpha}u_t, \quad t > 0, \quad 0 < x < a$$

$$\text{IC}: \qquad t = 0, \quad 0 \le x \le a, \quad u(x,0) = 0$$

$$\text{BCs}: \qquad t > 0, \quad u(0,t) = u_0, \qquad u_x(a,t) = 0$$

Laplace transformed equations:

$$\text{ODE}: \qquad \frac{d^2\bar{u}}{dx^2} = \frac{1}{\alpha}\left[s\bar{u} - u(x,0)\right], \quad 0 < x < a$$

Apply IC: $u(x,0) = 0$ to give

$$\text{ODE}: \qquad \frac{d^2\bar{u}}{dx^2} - \frac{s}{\alpha}\bar{u} = 0, \quad 0 < x < a$$

Transformed BCs are:

$$\bar{u}(0,s) = \frac{u_0}{s}, \qquad \frac{d\bar{u}(a,s)}{dx} = 0$$

Solutions in the $s-$ and $t-$domains are:

$$\bar{u}(x,s) = u_0 \left[\frac{\cosh \sqrt{s/\alpha}\,(a-x)}{s \cosh \sqrt{s/\alpha}\; a} \right], \quad s > 0$$

Therefore the solution of the one-dimensional diffusion equation in a finite domain is

$$\frac{u(x,t)}{u_0} = 1 + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)}\, e^{-(2n-1)^2 \frac{\pi}{4}^2 \frac{\alpha t}{a^2}} \cos(2n-1)\frac{\pi}{2}\frac{(a-x)}{a}$$

Another equivalent form of the solution is:

$$\frac{u(x,t)}{u_0} = 1 - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)}\, e^{-(2n-1)\frac{\pi}{4}^2 \tau} \sin(2n-1)\frac{\pi}{2}\xi$$

with $\tau = \alpha t/a^2 > 0$ and $0 \leq \xi = x/a \leq 1$, the dimensionless time and dimensionless position respectively.

Lecture 4

• Makeup Lecture
• Problem of last lecture was reformulated. Change a to L and reverse the location of the two boundary conditions. The physics of the problem is unchanged. The new problem formulation is

$$u_{xx} = \frac{1}{\alpha} u_t, \quad t > 0, \quad 0 < x < L$$

with initial condition:

$$t = 0, \quad 0 \leq x \leq L, \quad u(x,0) = 0$$

and boundary conditions:

$$t > 0, \quad x = 0, \quad u_x = 0, \quad \text{and} \quad x = L, \quad u = u_0 \quad \text{(constant)}$$

The Laplace tranformed equation is identical to previous formulation. The Laplace transform of the BCs gives:

$$\frac{\bar{u}(0,s)}{dx} = C_2\sqrt{\frac{s}{\alpha}} = 0 \quad \text{which requires} \quad C_2 = 0$$

and

$$\bar{u}(L,s) = C_1 \cosh\sqrt{\frac{s}{\alpha}}L = \frac{u_0}{s} \quad \text{which requires} \quad C_1 = \frac{u_0}{s\cosh\sqrt{\frac{s}{\alpha}}L}$$

The solution in the $s-$domain is

$$\bar{u}(x,s) = \frac{u_0 \cosh\sqrt{\frac{s}{\alpha}}x}{s\cosh\sqrt{\frac{s}{\alpha}}L}, \quad s > 0, \quad x \geq 0$$

The solution in the $t-$domain is

$$u(x,t) = \mathcal{L}^{-1}\{\bar{u}(x,s)\}$$

Using entry from Laplace transform table: No. 32.153 on page 171 of Schaum's Outline, by comparison of the terms we find:

$$a = \frac{L}{\sqrt{\alpha}} \quad \text{and} \quad x = \frac{x}{\sqrt{\alpha}}$$

The solution is

$$\frac{u(x,t)}{u_0} = 1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}\, e^{-(2n-1)^2\frac{\pi}{4}^2\frac{\alpha t}{L^2}}\cos(2n-1)\frac{\pi}{2}\frac{x}{L}, \quad t > 0, \quad 0 \leq x \leq L$$

• Dimensionless variables:

$$\phi = \frac{u(x,t)}{u_0} = f(\xi,\tau), \quad \text{where} \quad \tau = \frac{\alpha t}{L^2} \quad \text{and} \quad \xi = \frac{x}{L}$$

• Discuss physics of the problem and its solution. The solution consists of the steady-state part represented by the constant 1, and the transient part respresented by the summation term.

• Show plots of the dimensionless solution $\phi(\xi,\tau)$: $0 \leq \phi \leq 1$ in $0 \leq \xi \leq 1$ for $\tau \geq 0$.

Lecture 5

• Laplace tranform method applied to 1D diffusion in half-space subjected to Robin boundary condition (RBC).

• Solution in $s-$domain

$$\bar{u}(x,s) = \frac{\dfrac{hu_f}{ks}}{\dfrac{h}{k} + \sqrt{\dfrac{s}{\alpha}}} e^{-\sqrt{\dfrac{s}{\alpha}}x}, \quad s > 0, \quad x \geq 0$$

To find the inverse Laplace transform, use the short table presented on the ME 303 Web site. The last entry of the table shows that the inversion of

$$f(s) = \frac{be^{-a\sqrt{s}}}{s(b+\sqrt{s})}$$

is

$$F(t) = -e^{ab}e^{b^2t}\, erfc\left(b\sqrt{t} + \frac{a}{2\sqrt{t}}\right) + erfc\left(\frac{a}{2\sqrt{t}}\right)$$

• Comparison of $e^{-a\sqrt{s}}$ and $e^{-\sqrt{s/\alpha}x}$ requires

$$a = \frac{x}{\sqrt{\alpha}}$$

Next compare the terms:

$$\frac{b}{s(b+\sqrt{s})} \quad \text{and} \quad \frac{\dfrac{h}{ks}}{\dfrac{h}{k} + \sqrt{\dfrac{s}{\alpha}}} = \frac{\dfrac{h\sqrt{\alpha}}{k}}{s\left(\dfrac{h\sqrt{\alpha}}{k} + \sqrt{s}\right)}$$

By comparison we find that

$$b = \frac{h\sqrt{\alpha}}{k}$$

Now we have the relations:

$$ab = \frac{hx}{k}, \quad b^2t = \frac{h^2}{k^2}\alpha t, \quad \frac{a}{2\sqrt{t}} = \frac{x}{2\sqrt{\alpha t}}$$

and

$$e^{ab}e^{b^2t} = e^{ab+b^2t} = \exp\left(\frac{hx}{k} + \frac{h^2}{k^2}\alpha t\right)$$

• Solution is

$$\frac{u(x,t)}{u_f} = erfc\left(\frac{x}{2\sqrt{\alpha t}}\right) - \exp\left(\frac{hx}{k} + \frac{h^2}{k^2}\alpha t\right)\ erfc\left(\frac{h}{k}\sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}}\right), \quad t > 0, \quad x \geq 0$$

where u_f is the hot fluid temperature, $h > 0$ is the heat transfer coefficient, $k > 0$ is the thermal conductivity, and $\alpha > 0$ is the thermal diffusivity.

• Compare the solution with that given in the handout which comes from the ME 353 Heat Transfer Text. Examine the plots of the solution.

• Surface Temperature $u(0,t)$

$$\frac{u(0,t)}{u_f} = 1 - \exp\left(\frac{h^2}{k^2}\alpha t\right)\ erfc\left(\frac{h}{k}\sqrt{\alpha t}\right)$$

which is a function of time.

• Surface Heat Flux $q_0(t)$
¿From Fourier's Law of Conduction

$$q_0 = -k\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0}$$

and with the help of Maple we obtain

$$q_0 = h\, u_f\ \exp\left(\frac{h^2}{k^2}\alpha t\right)\ erfc\left(\frac{h}{k}\sqrt{\alpha t}\right)$$

which is also a function of time.

• Final Exam, Spring 1995, Question 1
– Question deals with fluid flow and conduction (convective heat transfer)
– (a) Check the units of left and right hand sides of PDE
– (b) Obtain the similarity parameter η given $\alpha = k/(\rho c_p)$
– (c) Transform PDE into ODE
– (d) Obtain the solution of ODE given the boundary conditions

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• This problem is similar to those covered in the course.
