Week 10

Lecture 1

• Discussed the physics of the problem of Project 2. Used Maple to show the temperature plots as a function of dimensionless time.
• Solution procedure is based on the material covered in Section 4 which deals with nonhomogeneous PDEs and nonhomogeneous BCs.
• Last tutorial, the TAs discussed this solution procedure as applied to the heat equation with nonhomgeneous BCs. The tutorial this week will consider the solution procedure applied to the nonhomogeneous wave equation.

• Section 2.3 Vibrations of beams: longitudinal (axial) and transverse.
• Longitudinal vibrations, $u(x,t)$:

$$u_{tt} = c^2 u_{xx}, \quad t > 0, \quad 0 < x < L, \quad c^2 = \frac{E}{\rho} > 0$$

where E is Young's modulus and ρ is the mass density. The PDE requires 2 BCs and 2 ICs for its solution. Separation of variables method (SVM) can be used.
• Transverse vibrations, $u(x,t)$:

$$u_{tt} + c^2 u_{xxxx} = 0, \quad t > 0, \quad 0 < x < L, \qquad c^2 = \frac{EI}{A\rho} > 0$$

where EI is the flexural rigidty, A is the cross-sectional area, and ρ is the mass density. The PDE is homogeneous. It requires 2 ICs and 4 BCs, two at each end, because this equation is fourth-order in x.

• Use SVM. Let $u(x,t) = X(x)T(t)$. One gets

$$\frac{T''}{T} + c^2 \frac{X^{iv}}{X} = 0 \quad \text{or} \quad \frac{T''}{c^2 T} + \frac{X^{iv}}{X} = 0$$

Choose the separation constant such that the time ODE is

$$T'' + c^2 \lambda^2 T = 0 \quad \text{and} \quad T(t) = C_1 \cos(\lambda c t) + C_2 \sin(\lambda c t)$$

The spatial ODE becomes:

$$X^{iv} - m^4 X = 0, \quad 0 < x < L$$

where $m^4 = \lambda^2$ for convenience. The solution is

$$X(x) = A\cos(mx) + B\sin(mx) + C\cosh(mx) + D\sinh(mx)$$

The four constants of integration: A, B, C, D can be found from 4 BCs specified at the ends: $x = 0$ and $x = L$. The boundary conditions are of the type:

$$u(0,t) = 0, \quad u_x(0,t) = 0, \quad u_{xx}(0,t) = 0, \quad u_{xxx}(0,t) = 0,$$

and

$$u(L,t) = 0, \quad u_x(L,t) = 0, \quad u_{xx}(L,t) = 0, \quad u_{xxx}(L,t) = 0$$

depending on the type of support such as i) simply supported, ii) built-in or fixed ends, and iii) free end. Consult your text on mechanics of deformable solids.

• Section 4.5 Heat Conduction With Radiation (Convective Cooling)
• A rod of constant cross-sectional area A, thermal conductivity k, and thermal diffusivity α. The temperature $u(x,t)$ is the solution of homogeneous PDE:

$$u_{xx} = \frac{1}{\alpha} u_t, \quad t > 0, \quad 0 < x < L$$

with IC:

$$t = 0, \quad 0 \le x \le L, \quad u(x,0) = f(x)$$

and BCs:

$$t > 0, \quad u(0,t) = 0, \quad \text{(homogeneous Dirichlet BC)}$$

and convective cooling at the end $x = L$, the solid-fluid interface.
• Derivation of Boundary Condition of the Third Kind (Robin Condition).
• Fourier's Law of Conduction on the solid side of the interface:

$$Q_{\text{cond}} = -kA \left(\frac{\partial u(x,t)}{\partial x} \right)_{x=L}$$

• Newton's Law of Cooling on the fluid side of the interface:

$$Q_{\text{conv}} = hA\left(u(L,t) - T_\infty\right)$$

where h is the heat transfer coefficient, and T_∞ is the ambient temperature. An energy balance (heat balance) at the end $x = L$ where $Q_{\text{cond}} = Q_{\text{conv}}$ gives the Robin BC:

$$\left(\frac{\partial u(x,t)}{\partial x}\right)_{x=L} = -\frac{h}{k}\left(u(L,t) - T_\infty\right)$$

This BC is nonhomogeneous due to T_∞. It can be made homogeneous by the introduction of $\theta(x,t) = u(x,t) - T_\infty$. Note that

$$\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial x}, \quad \text{because} \quad \frac{\partial T_\infty}{\partial x} = 0$$

The Robin condition becomes:

$$\left(\frac{\partial \theta(x,t)}{\partial x}\right)_{x=L} = -\frac{h}{k}\theta(L)$$

which is homogeneous.

• Observe that Spiegel has used the symbol h to represent the two parameters h/k. Many math texts do this. Be careful of the units.

Lecture 2

• Alternative formulation of the problem of Section 4.5.

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}, \quad t > 0, \quad 0 < x < L$$

$$\text{BCs}: \quad x = 0, \quad \theta(0,t) = 0 \quad \text{and} \quad x = L, \quad \frac{\partial \theta}{\partial x} = -\frac{h}{k}\theta$$

and

$$\text{IC}: \quad t = 0, \quad 0 \le x \le L, \quad \theta(x,0) = u(x,0) - T_\infty = f(x) - T_\infty$$

• Solution is

$$\theta(x,t) = T(t)X(x) = e^{-\lambda^2 \alpha t}\left[C_1 \cos \lambda x + C_2 \sin \lambda x\right]$$

• Homogeneous BCs require:

$$X(0) = 0 \quad \text{and} \quad X'(L) + \frac{h}{k}X(L) = 0$$

The first BC at $x = 0$ requires that $C_1 = 0$, thus removing the cosine function from the solution. The second BC at $x = L$ requires:

$$C_2 \lambda \cos \lambda L + \frac{h}{k} C_2 \sin \lambda L = 0$$

Since $C_2 = 0$ gives a trivial solution we choose

$$\lambda \cos \lambda L + \frac{h}{k} \sin \lambda L = 0 \quad \text{or} \quad \delta \cos \delta + Bi \sin \delta = 0$$

where $\delta = \lambda L$ and $Bi = hL/k$ with $0 < Bi < \infty$. See Eq. (68) of the text. Recall that Spiegel's h in Eq. (68) is h/k of this formulation.

• Roots or zeros of the characteristic equation (CE).
The CE has an infinite set of roots for any value of Bi. The roots are ordered:

$$\delta_1 < \delta_2 < \delta_3 < \cdots < \delta_n < \delta_{n+1} < \cdots < \infty$$

Also the difference between consecutive roots have the property

$$\delta_{n+1} - \delta_n \to \pi \quad \text{as} \quad n \to \infty$$

The roots can be calculated by means of the Newton-Raphson iterative method. The location of the roots must be determined. Plots of CE can reveal the locations.

• Limiting values of the roots for $Bi = \infty$ and $Bi = 0$

$$Bi = \infty, \quad \sin \delta = 0, \quad \delta_n = n\pi, \quad n = 1, 2, 3, \ldots$$

where the zero root has been rejected because it gives a trivial solution.

$$Bi = 0, \quad \cos \delta = 0, \quad \delta_n = (2n - 1)\frac{\pi}{2}, \quad n = 1, 2, 3, \ldots$$

• Ranges of the roots of CE for $0 < Bi < \infty$
$\pi/2 < \delta_1 < \pi$
$3\pi/2 < \delta_2 < 2\pi$
$5\pi/2 < \delta_3 < 3\pi$
$(2n - 1)\pi/2 < \delta_n < n\pi$
Used Maple to locate the roots, and to calculate the roots.

• The solution is

$$\theta(x, t) = \sum_{n=1}^{\infty} D_n e^{-\lambda_n^2 \alpha t} \sin \lambda_n x, \quad t > 0, \quad 0 < x < L$$

where $\lambda_n = \delta_n / L$ are the roots of the CE. The initial condition requires

$$\theta(x, 0) = f(x) - T_\infty = \sum_{n=1}^{\infty} D_n \sin \lambda_n x, \quad 0 < x < L$$

This is a Fourier sine series. The Fourier coefficients D_n can be found by means of the orthogonality property of the sines. Multipy the left-hand side and all terms of the right-hand side by $\sin \lambda_m x \, dx$ and integrate with respect to x. Thus

$$\int_0^L (f(x) - T_\infty) \sin \lambda_m x \, dx = \sum_{n=1}^{\infty} D_n \int_0^L \sin \lambda_m x \sin \lambda_n x \, dx$$

When $\lambda_m \neq \lambda_n$ we have

$$\int_0^L \sin \lambda_m x \sin \lambda_n x \, dx = 0$$

otherwise when $\lambda_m = \lambda_n$,

$$\int_0^L \sin^2 \lambda_n x \, dx = \frac{\lambda_n L - \cos \lambda_n L \sin \lambda_n L}{2\lambda_n} = \frac{L}{2} \left[\frac{\delta_n - \cos \delta_n \sin \delta_n}{\delta_n} \right]$$

Using the relationship given by the CE, the above integral can be written as

$$\int_0^L \sin^2 \lambda_n x \, dx = \frac{L}{2} \frac{Bi + \cos^2 \delta_n}{Bi}$$

and the Fourier coefficients are given by

$$D_n = \frac{Bi}{Bi + \cos^2 \delta_n} \frac{2}{L} \int_0^L (f(x) - T_\infty) \sin \lambda_n x \, dx$$

To proceed further one needs to specify the function $f(x)$.

Lecture 3

• Similarity Method (Boltzmann Transformation)
Transforms PDE into ODE.

One-dimensional diffusion equation:

$$T_{xx} = \frac{1}{\alpha} T_t, \quad t > 0, \quad x > 0$$

with initial condition:

$$T(x, 0) = T_i, \quad x \geq 0$$

and boundary conditions for $t > 0$:

$$T(0, t) = T_0, \qquad T(x \to \infty, t) \to T_i$$

Define temperature excess to create homogeneous conditions: $\theta(x,t) = T(x,t) - T_i$. The problem becomes:

$$\theta_{xx} = \frac{1}{\alpha}\theta_t, \quad t > 0, \quad x > 0$$

with initial condition:

$$\theta(x,0) = 0, \quad x \geq 0$$

and boundary conditions for $t > 0$:

$$\theta(0,t) = \theta_0 = T_0 - T_i, \qquad \theta(x \to \infty, t) \to 0$$

- Introduce the similarity parameter: $\eta = \dfrac{x}{\sqrt{4\alpha t}}$. Note that

$$\frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{4\alpha t}} \quad \text{and} \quad \frac{\partial \eta}{\partial t} = \frac{x}{\sqrt{4\alpha}}\left(-\frac{1}{2}t^{-3/2}\right)$$

- Let $\phi(\eta) = \theta(x,t)/\theta_0$.
- Transform the partial derivatives.

$$\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial \eta}\left[\theta_0 \phi\right]\frac{\partial \eta}{\partial x} = \frac{\theta_0}{\sqrt{4\alpha t}}\frac{\partial \phi}{\partial \eta}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial \eta}\left\{\frac{\theta_0}{\sqrt{4\alpha t}}\frac{\partial \phi}{\partial \eta}\right\}\frac{\partial \eta}{\partial x} = \frac{\theta_0}{4\alpha t}\frac{\partial^2 \phi}{\partial \eta^2}$$

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial \eta}\left(\theta_0\ \phi\right)\frac{\partial \eta}{\partial t} = \theta_0 \frac{\partial \phi}{\partial \eta}\frac{\partial}{\partial t}\left\{\frac{x}{\sqrt{4\alpha}}t^{-1/2}\right\} = \frac{x\ \theta_0}{\sqrt{4\alpha}}\left\{-\frac{1}{2}t^{-3/2}\right\}\frac{\partial \phi}{\partial \eta} =$$

$$-\frac{x\ \theta_0}{2t\sqrt{4\alpha t}}\frac{\partial \phi}{\partial \eta} = -\frac{\eta\ \theta_0}{2t}\frac{\partial \phi}{\partial \eta}$$

- Substitute into the PDE and replace

$$\frac{\partial^2 \phi}{\partial \eta^2} \quad \text{by} \quad \frac{d^2 \phi}{d\eta^2} \quad \text{because} \quad \phi(\eta) \quad \text{only}$$

$$\frac{\partial^2 \theta}{\partial x^2} - \frac{1}{\alpha}\frac{\partial \theta}{\partial t} = \frac{\theta_0}{4\alpha t}\frac{d^2 \phi}{d\eta^2} + \frac{\theta_0 \eta}{2\alpha t}\frac{d\phi}{d\eta} = 0$$

- Divide by θ_0 and multiply by $4\alpha t$ to get ODE:

$$\frac{d^2 \phi}{d\eta^2} + 2\eta\frac{d\phi}{\eta} = 0$$

• Transformed initial and boundary conditions:

$$t = 0, \quad \eta = \infty, \quad \phi = 0$$

and

$$t > 0, \quad x = 0, \quad \eta = 0, \quad \phi = 1$$

$$t > 0, \quad x \to \infty, \quad \eta \to \infty, \quad \phi \to 0$$

• Solution of ODE.
Let $w = d\phi/d\eta$ to reduce order of ODE.

$$\frac{dw}{d\eta} + 2\eta w = 0$$

• Apply separation of variables method to find the solution of the ODE. Therefore,

$$\frac{dw}{w} = -2\eta d\eta$$

After integration we get

$$w = \frac{d\phi}{d\eta} = C_1 e^{-\eta^2}$$

Another integration gives

$$\phi = C_1 \int_0^\eta e^{-\eta^2} d\eta + C_2$$

where lower limit was arbitrarily set to zero.
• Apply the boundary conditions to find the constants: C_1, C_2. When $\eta = 0$, $\phi = 1$, the integral is zero, and $C_2 = 1$.
When $\eta = \infty$, $\phi = 0$ and we have

$$0 = C_1 \int_0^\infty e^{-\eta^2} d\eta + 1 = C_1 \frac{\sqrt{\pi}}{2} + 1$$

The value of the integral is $\sqrt{\pi}/2$. The first constant of integration is

$$C_1 = -\frac{2}{\sqrt{\pi}}$$

• The solution of the ODE and therefore the PDE is

$$\phi = 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta^2} d\eta = 1 - erf(\eta) = erfc(\eta)$$

where $erf(\eta)$ and $erfc(\eta)$ are the error and complementary error functions with similarity parameter: $\eta = x/(2\sqrt{\alpha t})$.

• The solution can also be expressed as

$$\frac{T(x,t) - T_i}{T_0 - T_i} = erfc\left(\frac{x}{2\sqrt{\alpha t}}\right) \qquad t > 0, \quad x \geq 0$$

• See Maple worksheets for Similarity Method and some characteristics of the error and complementary error functions.
Some properties of these important special functions:
$erf(0) = 0, \quad erf(\infty) = 1, \quad erf(-\eta) = -erf(\eta), \quad erfc(\eta) = 1 - erf(\eta)$