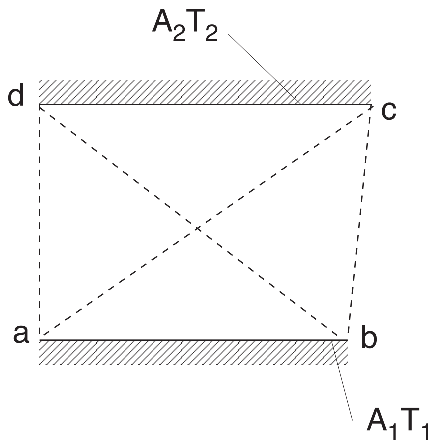
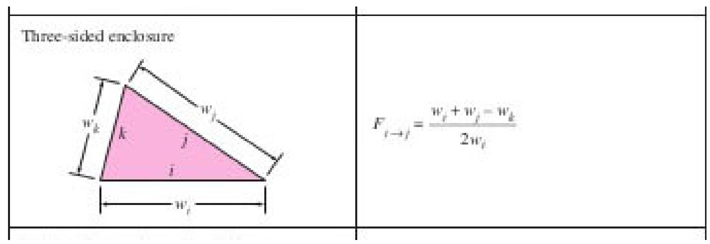
Hottel Crossed String Method

Can be applied to 2D problems where surfaces are any shape, flat, concave or convex. Note for a 2D surface the area, A is given as a length times a unit width.



$\Rightarrow$



$$A_1 F_{12} = A_2 F_{21} = \frac{\text{(total crossed)} - \text{(total uncrossed)}}{2}$$

A_1 and A_2 <u>do not have to be parallel</u>

$$A_1 F_{12} = A_2 F_{21} = \frac{1}{2}[\underbrace{(ac + bd)}_{crossed} - \underbrace{(bc + ad)}_{uncrossed}]$$

Proof:

$$A_1 F_{1bd} = \frac{ab + bd - ad}{2} \qquad \text{where} \quad A_1 = ab \times 1 \ \text{(unit depth)}$$

$$A_1 F_{1ad} = \frac{ab + ad - bd}{2}$$

$$A_1 F_{1ac} = \frac{ab + ac - bc}{2}$$

$$A_1 F_{1bc} = \frac{ab + bc - ac}{2}$$

but for an enclosure we have

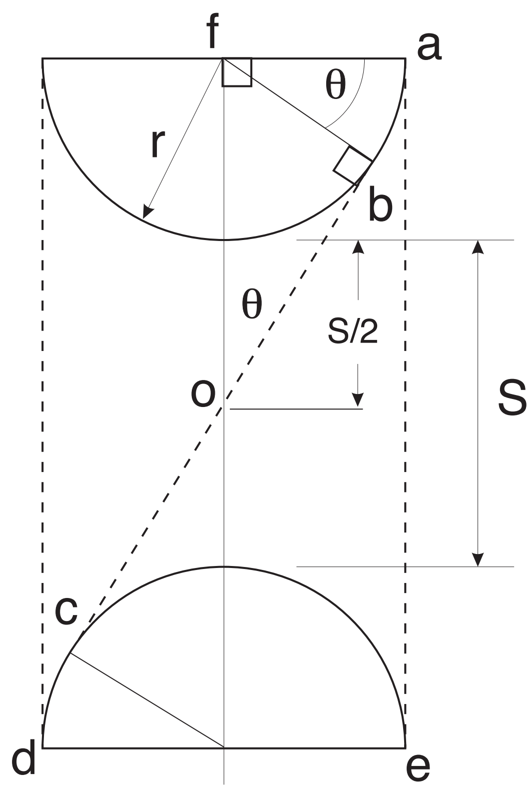
$$A_1 F_{1ad} + A_1 F_{1bc} + A_1 F_{12} = A_1$$

$$A_1 F_{12} = A_1 - \{A_1 F_{1ad} + A_1 F_{1bc}\} \qquad (1)$$

Substituting F_{1ad} and F_{1bc} into Eq. (1) yields

$$\begin{aligned} A_1 F_{12} &= ab - \left\{ \frac{ab + ad - bd}{2} + \frac{ab + bc - ac}{2} \right\} \\ &= \frac{2ab - ab - ad + bd - ab - bc + ac}{2} \\ &= \frac{(ac + bd) - (bc + ad)}{2} \quad \Leftarrow \text{ cross strings formula} \end{aligned}$$

Two Half Cylinders



Background

1. **Arc Length:** The arc length of a sector is given as $s = r\theta$
2. **Pythagorean Theorem:** $fo^2 = fb^2 + bo^2$
3. **Complimentary Angle Theorem:** $\angle\, afb = \angle fob$ since $\angle afo = 90^o$ and the two acute angles in triangle fbo also equal 90^o

By symmetry

$$\begin{aligned}
ab &= cd = r\theta \quad \text{(arc length and symmetry)} \\
\theta &= \sin^{-1}\frac{fb}{fo} = \sin^{-1}\frac{r}{r+s/2} = \sin^{-1}\frac{1}{1+\dfrac{s}{2r}} \\
bo &= oc = \sqrt{(fo)^2-(fb)^2} \quad \text{(Pythagorean theorm)} \\
&= \sqrt{\left(r+\frac{s}{2}\right)^2 - r^2} \\
&= r\sqrt{\left(\underbrace{1+\frac{s}{2r}}_{\equiv x}\right)^2 - 1} \\
&= r\sqrt{x^2-1}
\end{aligned}$$

Let $x = 1 + s/(2r)$; $\theta = \sin^{-1} 1/x$;

$$ae = s + 2r = 2r\left(1+\frac{s}{2r}\right) = 2rx$$

Therefore

$$A_1F_{12} = A_2F_{21} = ad - ae = \underbrace{\overbrace{2r\sqrt{x^2-1}}^{2bo} + \overbrace{2r\sin^{-1}\frac{1}{x}}^{2r\theta}}_{ad} - \underbrace{2rx}_{ae}$$

Since $A_1 = \pi r$ (from tight strings across cylinder)

$$F_{12} = \frac{2}{\pi}\left\{\sqrt{x^2-1} + \sin^{-1}\frac{1}{x} - x\right\}$$

where

$$x = 1 + \frac{s}{2r}$$